Preferred Habitats Across the Real and Nominal Term Structure of Interest Rates

Nimar C. Zhao*
London School of Economics

December 1, 2024 (Draft, latest version here)

Abstract. We develop a tractable model of the nominal and real term structure of interest rates based on the modern preferred habitat framework developed by Vayanos and Vila (2021). In our model, risk-averse arbitrageurs work to integrate both nominal and real yield curves over localised supply, while balancing their inflation and nominal short rate risk. Our model implies localised demand and supply between investors with preferences for specific maturities in one bond type can affect the equilibrium term structure of both inflation-protected and nominal assets. In order to evaluate our model empirically, we use a recursive vector auto-regression to identify shocks to model factors. Consistent with our theoretical predictions, we find evidence that cross-asset supply effects are positive and significant for real supply as measured by our theoretically motivated factor, maturity-weighted-debt-to-GDP. In addition, the estimated coefficients for inflation and nominal short rates closely match their calibrated model counterparts.

*Department of Economics, London School of Economics (LSE); e-mail: `n.c.zhao@lse.ac.uk`, and Mathematical Institute, University of Oxford; email: `nimar.zhao@maths.ox.ac.uk`. I am grateful to Kevin Zhao, Dimitri Vayanos, Walker D. Ray for their guidance and extensive discussions. I am grateful to Qiang Dai, Kevin D. Sheedy, Sitong Ding, Matthew Levy and other EC331 seminar participants for helpful comments. I am deeply indebted to Junkai Sun for his programming expertise.

1 Introduction

In what ways are the real and nominal term structure of interest rates linked? More often than not, the standard approach that links real with nominal interest rates at any given maturity is through their difference, known as break-even inflation. Intuitively, break-even inflation approximately represents market expectations of average inflation over that same maturity. For policymakers, break-even inflation therefore provides an important additional measure of long-run inflation that is significantly more frequently observed compared with traditional survey based approaches such as those from the Survey of Professional Forecasters (SPF) and the Michigan Consumer Survey. Long-run expectations are of central importance to monetary policymakers and pertinent in modern macroeconomic literature to ensuring the effectiveness of monetary policy. However, break-even inflation has sometimes yielded puzzling results and some of the largest documented arbitrage opportunities (Mishkin and Wright 2009; Fleckenstein et al. 2010), that have either been attributed to liquidity premiums, or market segmentation.

Our paper attempts to formalise the latter in a model based on the preferred habitat theory of the term structure, first proposed by Culbertson (1957) and Modigliani and Sutch (1966). By extending the modern preferred habitat framework of Vayanos and Vila (2021) to a joint model of the nominal and real term structure, we derive an affine term structure model (ATSM) that provides explicit microeconomic-foundations for demand and supply in both markets to jointly determine the equilibrium real and nominal term structure, as well as incorporating the more traditional factors such as inflation and short-term interest rates. In Section 2, we present our theoretical model and study its analytical properties. We assume inflation and interest rates follow an exogenous, mean-reverting process. Bond prices are determined endogenously by arbitrageur preferences and the available supply of bonds. We finish the theoretical section by deriving analytical properties of the model, and calibrate model parameters to form our theoretical predictions for yield responses to model factors.

We test the predictions of our model using data on the US Treasury market from 2001 to 2023. In Section 3 we construct our primary measure of bond supply, a historical time series of the maturity-weighted-debt-to-GDP of US debt. In Section 4, we identify shocks to our macroeconomic factors, and compare the empirical response to that implied by our model. Our model is able to account for changes in yields due to traditional macroeconomic factors effectively, as well as shifts in our supply measure, MWD-to-GDP. In particular, a shock to maturity-weighted-debt-to-GDP by one unit increases 20-yr nominal yields by around 305bps, and 20-yr real yields by 116bps. We consider a range of specifications and robustness tests and show our results are not significantly unaltered by these changes.. Finally, we subject our results to a number of robustness tests, by considering additional confounding factors, as well as different identifying assumptions.

1.1 Inflation-indexed bonds

Nominal bonds do not provide safety in real terms because expected inflation fluctuates over time. Instead, inflation-linked bonds (ILBs), known as Treasury Inflation Protected Securities (TIPS) in the United States, are bonds that protect against inflation by paying investors an inflation-adjusted coupon and principal based on the Consumer Price Index (specifically CPI-U). In the years immediately after their 1997 launch, the TIPS market experienced very poor liquidity, and at times there was talk that TIPS issuance would be discontinued (Gürkaynak and Wright, 2012). TIPS liquidity has improved over the subsequent years and now represents a significant fraction of marketable debt (see fig. 4) but despite this, TIPS liquidity sharply fell during the financial crisis (Campbell et al., 2009).

A few papers have attempted to model the nominal and real term structure jointly. Wachter (2006) models the term structure following a consumption-based approach, and when calibrated, the model is able to account for many short-run fluctuations in yields using only consumption and inflation data. Buraschi and Jiltsov (2005) estimate a real business cycle model with affine state variables that generate time varying inflation risk premia. They find this variable is important in explaining deviations from the Expectations Hypothesis. D'Amico et al. (2008) estimate a three factor no-arbitrage model and find a large, time-varying liquidity premium in TIPS yields; in addition, they conclude that TIPS implied break-even inflation rates are good indicators of actual inflation expectation as they find high frequency variation in the data coincides with model implied variation in inflation expectations. In a similar spirit, Christensen et al. (2010) estimate an arbitrage-free Nelson-Siegel model of the term structure and also find that long-term inflation expectations have been well anchored over the past few years, and well proxied by TIPS implied break-even inflation rates. Piazzesi and Schneider (2006) model how inflation negatively impacts nominal bonds not only in real terms, but also through a worsening outlook on consumption growth, however their model does not account for positive term premia.

1.2 Preferred habitats

The Vayanos and Vila (2021) preferred habitat approach to modelling the yield curve is relatively new in the class of ATSM, but has the unique advantage of providing explicit relationships between yields and supply. Traditional theories of the term structure (Cox et al. 1985; Vasicek 1977) do not model local demand or supply in the yield curve, however price pressure in bond markets have been used to explain various phenomena in yield movements.

Historical events include the Operation Twist program, carried out by the U.S. Treasury and the Federal Reserve from 1962 to 1964, aimed at reducing the average maturity of government debt (e.g. Modigliani and Sutch 1966; Swanson 2011), and the U.S. Treasury's buyback program during 2000-2002 for a similar goal (e.g. Garbade and Rutherford 2007; Bernanke et al. 2004; Greenwood and Vayanos 2014). More recently, the United States has engaged in large scale quantitative easing (QE) and tightening (QT) programs (e.g. Krishnamurthy and Vissing-Jorgensen 2012; D'Amico and King 2013; Gorodnichenko et al. 2023).

The papers aforementioned are typically silent on the macroeconomic implications of preferred habitat theory. Conversely, advancements in macroeconomic theory (e.g. Gertler and Karadi 2011; Cúrdia and Woodford 2011), focus on aggregate variables yet fail to capture reflect the complex dynamics observed across the term structure. Ray (2019) and Gorodnichenko et al. (2023) connect both sides in a New-Keynesian model with an embedded preferred habitat model of the term structure, which provides valuable insight into the macro-financial dynamics of QE. Despite their more realistic inflation dynamics with jumps, the model is intractable and its properties must be verified by numerical methods. We therefore propose our tractable 'reduced form' model of inflation and interest rates to focus explicitly on the relationship between inflation-linked bonds and nominal bonds.

2 Theoretical Predictions

In this section, we develop a theoretical model of nominal and inflation-indexed bonds, building on the seminal work by Vayanos and Vila (2021) in which arbitrageurs integrate the term structure across localised demand and supply for bonds of different maturities - rendering both term structures arbitrage-free. Their ability to do so is constrained by their degree of risk-aversion, a. For tractability, our model is set in continuous time, focusing on a two-factor version of Vayanos and Vila (2021) with deterministic changes in supply. This allows us to derive closed-form solutions for bond prices, as well as their response to macroeconomic and supply factors, which we take to the data in the following sections.

2.1 Macroeconomic factors

We first describe the evolution of macroeconomic factors. The aggregate price index Π_t is related to the inflation rate in differential form,

$$\mathrm{d}\log \Pi_t = \pi_t\,\mathrm{d}t; \qquad \Pi_0 = 1 \tag{1}$$

We denote the instantaneous nominal interest rate by i_t, and the instantaneous real interest rate by $r_t := i_t - \pi_t$. Our factors $\mathbf{q}_t = (\pi_t, i_t)^\top$ follow the exogenous Ornstein-Uhlenbeck process,

$$\mathrm{d}\mathbf{q}_t = \mathbf{\Gamma}\,(\bar{\mathbf{q}} - \mathbf{q}_t)\,\mathrm{d}t + \mathbf{\Sigma}\,\mathrm{d}\mathbf{B}_t \tag{2}$$

with parameter matrices $\mathbf{\Gamma}, \mathbf{\Sigma} \in \mathbb{R}^{2\times 2}$ defined by,

$$\mathbf{\Gamma} := \begin{pmatrix} \kappa_\pi & \gamma \\ -\phi & \kappa_i \end{pmatrix}, \qquad \mathbf{\Sigma} := \begin{pmatrix} \sigma_\pi & \sigma_{\pi i} \\ \sigma_{i\pi} & \sigma_i \end{pmatrix} \tag{3}$$

long-run constants $\bar{\mathbf{q}} := (\bar{\pi},\, \bar{\imath})^\top \in \mathbb{R}^2$, and a vector of independent Brownian motions $\mathbf{B}_t := (B_{\pi t},\, B_{it})^\top$ with respect to the natural filtration. In this specification, inflation and nominal short rates follow potentially correlated processes; assuming all the scalar parameters are positive implies inflation responds negatively to interest rates, while interest rates respond positively to inflation in a Taylor (1993) style rule. The coefficients $\sigma_i, \sigma_\pi, \sigma_{i,\pi}$ and $\sigma_{\pi,i}$ represent the sensitivity of the factors to the respective Brownian terms.

2.2 Bond prices and yields

The real and nominal term structure at time t consists of a continuum of zero-coupon bonds with maturities in the interval $(0, T]$. Nominal bonds, with subscript n pay one nominal unit at maturity, whereas inflation-linked bonds, with subscript r pay the level of the inflation-index, Π_t at maturity. This is a minor modification of the actual TIPS payoff structure, which scales down the payoff by the inflation-index level at issuance, but retains identical definitions for yields. We also note that actual TIPS payoffs are also bounded below by the nominal payoff; we omit this inflation optionality feature to preserve the tractability of our model. The true TIPS bond would instead slightly dominate the type of inflation-linked bond in our model.

Throughout, we maintain that variables are expressed in real units, while nominal variables are made distinct with a tilde. *Nominal bonds* have nominal price $\tilde{P}^{(\tau)}_{nt}$ with nominal yield $\tilde{y}^{(\tau)}_{nt}$, as well as a real price $P^{(\tau)}_{nt} := \tilde{P}^{(\tau)}_{nt} / \Pi_t$ with real yield $y^{(\tau)}_{nt}$ satisfying,

$$\tilde{P}^{(\tau)}_{nt} = \exp(-\tau\,\tilde{y}^{(\tau)}_{nt}) \tag{4}$$

$$P^{(\tau)}_{nt} = \exp(-\tau\,y^{(\tau)}_{nt})\,\mathbb{E}_t[\Pi_{t+\tau}]^{-1} \tag{5}$$

On the other hand, *inflation-linked bonds* have nominal price $\tilde{P}^{(\tau)}_{rt}$ with nominal yield $\tilde{y}^{(\tau)}_{rt}$, as well as real price $P^{(\tau)}_{rt} := \tilde{P}^{(\tau)}_{rt} / \Pi_t$ with real yield $y^{(\tau)}_{rt}$ satisfying,

$$\tilde{P}^{(\tau)}_{rt} = \exp(-\tau\,\tilde{y}^{(\tau)}_{rt})\,\mathbb{E}_t[\Pi_{t+\tau}] \tag{6}$$

$$P^{(\tau)}_{rt} = \exp(-\tau\,y^{(\tau)}_{rt}) \tag{7}$$

We maintain the convention that 'nominal' yields refer to the nominal yield on the nominal bond, while 'real yields' refer to the real yield on the inflation-linked bond.

2.3 Arbitrageurs

Bonds are issued by a government and are traded by arbitrageurs. As in Vayanos and Vila (2021); Ray et al. (2024), arbitrageurs choose a sequence of portfolios $(x_{nt}^{(\tau)}, x_{rt}^{(\tau)})_{\tau\in(0,T]}$ consisting of nominal and inflation-linked bonds that maximises mean-variance utility over instantaneous changes in real wealth, $\mathrm{d}W_t$. They face the problem,

$$\max_{\left(x_{nt}^{(\tau)}, x_{rt}^{(\tau)}\right)_{\tau\in(0,T]}} \left[\mathbb{E}_t[\mathrm{d}W_t] - \frac{a}{2}\operatorname{var}_t(\mathrm{d}W_t)\right] \tag{8}$$

$$\text{s.t.} \quad \mathrm{d}W_t = W_t r_t\,\mathrm{d}t + \int_0^T \sum_{j=n,r} x_{jt}^{(\tau)} \left(\frac{\mathrm{d}P_{jt}^{(\tau)}}{P_{jt}^{(\tau)}} - r_t\,\mathrm{d}t\right)\mathrm{d}\tau \tag{9}$$

where $a \geq 0$ is the coefficient of risk aversion that captures the trade-off between higher expected returns and portfolio risk, and $x_{jt}^{(\tau)}$ is the real value of positions in bond type j. We can interpret these arbitrageurs as overlapping generations of short-lived agents with von Neumann-Morgenstern preferences that start with the same level of wealth W_t, end with wealth $W_{t+\mathrm{d}t}$ at time $t + \mathrm{d}t$ and have an absolute risk-aversion coefficient of a.

2.4 Equilibrium term structure

We solve for an equilibrium term structure in three steps: (1) we conjecture a functional form for equilibrium yields, (2) given the conjecture, derive the arbitrageur's optimality conditions and (3) impose market clearing conditions to verify the conjectured form of bond prices. As with a standard Vasicek model, we focus on (exponential) affine forms for bond prices; that is, there exists four functions $\mathbf{A}_j : (0,T] \to \mathbb{R}^2, C_j : (0,T] \to \mathbb{R}$ for $j \in \{n,r\}$ that depend solely on maturity τ, satisfying the following relations: for nominal bonds, $j = n$ we conjecture that log *nominal* prices are affine in $\mathbf{q}_t$,

$$-\log \tilde{P}_{nt}^{(\tau)} = \mathbf{A}_n(\tau)^\top \mathbf{q}_t + C_n(\tau) \tag{10}$$

while for inflation linked bonds, we conjecture that log *real* prices are affine in $\mathbf{q}_t$,

$$-\log P_{rt}^{(\tau)} = \mathbf{A}_r(\tau)^\top \mathbf{q}_t + C_r(\tau) \tag{11}$$

Applying Itô's lemma to our conjectures, real returns for bond type j are given by

$$\frac{\mathrm{d}P_{jt}^{(\tau)}}{P_{jt}^{(\tau)}} = \mu_{jt}^{(\tau)} - \mathbf{A}_j(\tau)^\top \mathbf{\Sigma}\,\mathrm{d}\mathbf{B}_t \tag{12}$$

where $\mu_{jt}^{(\tau)}$ is the instantaneous expected real return defined by

$$\mu_{jt}^{(\tau)} := \mathbf{A}_j'(\tau)^\top \mathbf{q}_t + C_j'(\tau) + \mathbf{A}_j(\tau)^\top \mathbf{\Gamma}\left(\mathbf{q}_t - \bar{\mathbf{q}}\right) + \frac{1}{2}\,\mathbf{A}_j(\tau)\,\mathbf{\Sigma}\mathbf{\Sigma}^\top \mathbf{A}_j(\tau) - \pi_t\,\mathbf{1}_{\{j=n\}} \tag{13}$$

and $\mathbf{1}_{\{j=n\}}$ is a standard indicator function. To derive the arbitrageur's first order condition, we proceed by substituting the expressions for bond returns into the arbitrageur's budget constraint as shown in the proof of Lemma 1.

Lemma 1. The arbitrageur's optimality condition for bond type j is

$$\mu_{jt}^{(\tau)} - r_t = -\mathbf{A}_j(\tau)^\top \boldsymbol{\lambda}_t \tag{14}$$

where $\boldsymbol{\lambda}_t \in \mathbb{R}^2$ is a vector of risk prices given by,

$$\boldsymbol{\lambda}_t := -a\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \left[\int_0^T x_{nt}^{(\tau)} \mathbf{A}_n(\tau) + x_{rt}^{(\tau)} \mathbf{A}_r(\tau) \, \mathrm{d}\tau \right] \tag{15}$$

The proof of Lemma 1 can be found in Appendix A. Lemma 1 states that the excess real return of bond type j, is a linear function of the bond's sensitivities $\mathbf{A}_j(\tau)$ to the risk factors in $\mathbf{q}$. The process $\boldsymbol{\lambda}_t$ therefore measures the excess return per unit sensitivity to each factor in $\mathbf{A}_j(\tau)$. While our model follows the preferred-habitat framework, (14) is common to the class of no-arbitrage affine term structure models (Duffie and Kan 1996; Dai and Singleton 2000), the broader class of ATSMs do not impose restrictions on how to form these risk prices, only on their existence. Our model instead derives risk prices by a market clearing argument, that is the market for each maturity τ must clear, which we present in the next section.

2.5 Market clearing

The portfolio that arbitrageurs hold in equilibrium is determined by the market portfolio. For a given $j \in \{n, r\}$ and $\tau \in (0, T]$, let $(s_{jt}^{(\tau)})_{t \in \mathbb{R}_+} \subseteq \mathbb{R}$ be a deterministic sequence of supply for bonds of maturity τ. Market clearing implies

$$x_{jt}^{(\tau)} = s_{jt}^{(\tau)} \tag{16}$$

which equates the arbitrageur's real investment in bond j with maturity τ to the bond's supply expressed in real units. Substituting the definition of $\mu_{jt}^{(\tau)}$ and market clearing into the arbitrageur's first order condition (14), we find that

$$\left[\mathbf{A}_j'(\tau)^\top \mathbf{q}_t + C_j'(\tau) + \mathbf{A}_j(\tau)^\top \boldsymbol{\Gamma}(\mathbf{q}_t - \bar{\mathbf{q}}) + \frac{1}{2}\mathbf{A}_j(\tau)\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \mathbf{A}_j(\tau) \right] - i_t + \pi_t \, \mathbf{1}_{\{j=r\}} = a\mathbf{A}_j(\tau)^\top \boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \left[\boldsymbol{\Delta}_{nt} + \boldsymbol{\Delta}_{rt} \right] \tag{17}$$

where $\boldsymbol{\Delta}_{jt} := \int_0^T s_{jt}^{(\tau)} \mathbf{A}_j(\tau) \, \mathrm{d}\tau$ is the dollar-duration of the supply portfolio for bond type j. Identifying linear terms in $\mathbf{q}_t$ yields the ODE system

$$\mathbf{A}_j'(\tau) + \boldsymbol{\Gamma}^\top \mathbf{A}_j(\tau) - \mathbf{c}_j = 0 \tag{18}$$

where $\mathbf{c}_j := (-\,\mathbf{1}_{\{j=r\}}, 1)^\top$ is a constant vector. (18) is solved with terminal condition $\mathbf{A}_j(0) = \mathbf{0}$; which follows from our affine conjecture as both bonds pay one at maturity, in nominal or real units respectively. Similarly for $C_j(\tau)$, identifying constant terms in (19) leaves

$$C_j'(\tau) = \mathbf{A}_j(\tau)^\top \boldsymbol{\Gamma}\bar{\mathbf{q}} - \frac{1}{2}\mathbf{A}_j(\tau)\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \mathbf{A}_j(\tau) + a\mathbf{A}_j(\tau)^\top \boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \left[\boldsymbol{\Delta}_{nt} + \boldsymbol{\Delta}_{rt} \right] \tag{19}$$

Having already solved for $\mathbf{A}_j(\tau)$ in (18), directly integrating with terminal condition $C_j(0) = 0$ yields the solution. The theorem below states the solution to both ODEs, and is proved in Appendix A.

Theorem 2. Define the components of $\mathbf{A}_j(\tau) := (A_{\pi j}(\tau), A_{ij}(\tau))^\top$; the solution to (18) can be shown as

$$A_{\pi j}(\tau) = -\frac{1-e^{-\nu_1 \tau}}{\nu_1}\mathbf{1}_{\{j=r\}} + \frac{\nu_2 - \kappa_i}{\nu_2 - \nu_1}\left(\frac{1-e^{-\nu_2 \tau}}{\nu_2} - \frac{1-e^{-\nu_1 \tau}}{\nu_1}\right)\mathbf{1}_{\{j=n\}} \tag{20}$$

$$A_{ij}(\tau) = \frac{1-e^{-\nu_1 \tau}}{\nu_1} - \frac{\nu_1 - \kappa_i}{\nu_2 - \nu_1}\left(\frac{1-e^{-\nu_2 \tau}}{\nu_2} - \frac{1-e^{-\nu_1 \tau}}{\nu_1}\right)\mathbf{1}_{\{j=n\}} \tag{21}$$

where $\nu_2 > \nu_1 > 0$ are the eigenvalues of $\mathbf{\Gamma}$ defined as

$$\nu_1 := \frac{\kappa_\pi + \kappa_i - \sqrt{(\kappa_\pi - \kappa_i)^2 - 4\phi\gamma}}{2}, \quad \nu_2 := \frac{\kappa_\pi + \kappa_i + \sqrt{(\kappa_\pi - \kappa_i)^2 - 4\phi\gamma}}{2} \tag{22}$$

The solution to (19) is then

$$C_j(\tau) = \left[\int_0^\tau \mathbf{A}_j(u)\,\mathrm{d}u\right]^\top \boldsymbol{\chi} - \frac{1}{2}\int_0^\tau \mathbf{A}_j(u)^\top \mathbf{\Sigma}\mathbf{\Sigma}^\top \mathbf{A}_j(u)\,\mathrm{d}u \tag{23}$$

where

$$\boldsymbol{\chi} := \mathbf{\Gamma}\bar{\mathbf{q}} + a\mathbf{\Sigma}\mathbf{\Sigma}^\top\left[\mathbf{\Delta}_{nt} + \mathbf{\Delta}_{rt}\right] \tag{24}$$

The proof of Theorem 2 is contained in Appendix A. We note that our solution is a linear transformation of the vector-valued function $\mathbf{g}(\tau) = (g_1(\tau), g_2(\tau))^\top$ with components $g_j(\tau) = \frac{1-e^{-\nu_j \tau}}{\nu_j}$. With this notation, our solution for $\mathbf{A}_j(\tau)$ can be expressed as a linear transformation of $\mathbf{g}(\tau)$. In our model, as arbitrageurs optimise over changes in real wealth, inflation-linked bonds behave as the nominal bonds in Vayanos and Vila (2021). By substituting our solution into the conjecture, we find that inflation-linked bonds are priced by the real short rate,

$$-\log P_{rt}^{(\tau)} = g_1(\tau)\, r_t + C_r(\tau) \tag{25}$$

With fixed supply, our model admits no-arbitrage due to the existence of factor risk-prices (?). In Appendix B, we show how the model can be extended to include a stochastic supply factor, which forms additional risk that the arbitrageurs must be compensated against. In a neighbourhood around risk-neutrality, an equilibrium solution exists and thus no-arbitrage holds.

2.6 Effects of factors

In this part, we begin by showing properties of the model analytically. In particular, we derive properties of how yields respond to the four factors in the model, that is the nominal short rate, inflation rate, nominal supply and inflation-indexed supply. For this part of our discussion, we focus on the case that eigenvalues ν_1 and ν_2 are real; Vayanos and Vila (2021) show that the complex case yields similar results within the first half-cycle of the oscillation. Let $\mathbf{y}_t^{(\tau)} = (\tilde{y}_{nt}^{(\tau)}, y_{rt}^{(\tau)})^\top \in \mathbb{R}^2$ stack 'nominal' and 'real' yields, and $\boldsymbol{\mathcal{A}}(\tau) = (\mathbf{A}_n(\tau), \mathbf{A}_r(\tau)) \in \mathbb{R}^{2\times 2}$ pack the affine coefficients.

Proposition 1. The response of yields $\mathbf{y}_t^{(\tau)}$ to the factor vector $\mathbf{q}_t$ is the 2×2 matrix given by

$$\frac{\partial \mathbf{y}_t^{(\tau)}}{\partial \mathbf{q}_t} = \tau^{-1}\boldsymbol{\mathcal{A}}(\tau) \tag{26}$$

Moreover, for

a. *inflation-linked bonds*, the response to

i. inflation is negative and increasing monotonically with limit zero as $\tau \to \infty$,

ii. short rates is positive and decreasing monotonically with limit zero as $\tau \to \infty$.

b. *nominal bonds*, if $\kappa_\pi - \kappa_i > 0$, the response to

i. inflation is negative, and has limit zero as $\tau \to \infty$,

ii. short rates is positive, and decreasing monotonically with limit zero as $\tau \to \infty$.

The proof of Proposition 1 is in Appendix A. Proposition 1 provides us functional forms for the response functions that are used to parameterise the model, as well as sign hypotheses for our yield response coefficients that we later estimate in the empirical section. To plot these functions, we calibrate the Ornstein-Uhlenbeck process in (2) using its discrete time analogue, the VAR(1) process; the details of this process are discussed in Appendix C. We then plot these functions in Figure 1 based on the calibrated parameters in Table C1.

For real bonds, shocks to the nominal short rate and inflation affect yields with the same magnitude, as they both affect the real short rate with equal size and opposite direction. The mechanism behind this is that arbitrageurs transmit real short rate shocks to bond yields through carry trades. Consider a negative shock to the real short rate that has not been 'priced in' to yields. To take advantage of this temporary mis-pricing, arbitrageurs would adjust their portfolio to buy long maturity bonds by borrowing at the real short rate. Their demand causes long maturity bonds to rise, and thus their yields to fall - adjusting yields down. On the other hand, for nominal bonds a positive inflation shock decreases yields through the effect on the real interest rate, but is offset by reduced demand as ILB payoff increases in nominal terms. Therefore, in Figure 1 we observe a small, negative response of nominal yields to inflation.

2.7 Effects of supply

The next proposition looks at the effect of an unanticipated, permanent ('MIT') shock to supply.

Proposition 2. The response of yields to a change in supply of bond $j \in \{n, r\}$ of maturity τ' at time t is

$$\frac{\partial \mathbf{y}_t^{(\tau)}}{\partial s_{jt}^{(\tau')}} = \frac{a}{\tau} \left[\int_0^\tau \boldsymbol{\mathcal{A}}(u)\, \mathrm{d}u \right]^\top \boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \mathbf{A}_j(\tau') \tag{27}$$

This proposition formalises the idea that local supply shocks, say for maturity τ' have effects on bonds prices across the whole yield curve, which Greenwood et al. (2023) term 'global price pressure'. For both types of bonds, supply effects enter our affine equilibrium through $C_j(\tau)$; in particular due to the quantity $\boldsymbol{\Delta}_t$, the dollar-duration of the supply portfolio. Although this effect is standard in single asset preferred habitat models such as Greenwood and Vayanos (2014); Vayanos and Vila (2021), in our model arbitrageurs must additionally optimise their portfolio across nominal and real bonds. This creates a link between the two markets that means nominal supply is relevant for real yields and vice versa.

In our empirical analysis, we focus on the quantity M_{jt} that denotes maturity-weighted debt - defined as $M_{jt} := \int_0^T s_{jt}^{(\tau)}\, \tau\, \mathrm{d}\tau$. With our calibration, we observe that the scalar components of $\boldsymbol{\Delta}_{jt}$, $\int_0^T s_{jt}^{(\tau)} A_{\pi j}(\tau)\, \mathrm{d}\tau$ and $\int_0^T s_{jt}^{(\tau)} A_{ij}(\tau)\, \mathrm{d}\tau$ have a -99.99% and 99.79% correlation with M_{jt} respectively; therefore we approximate $\boldsymbol{\Delta}_t$ with $\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) M_{jt}$. The supply factor in our model can be therefore be approximated by

$$\boldsymbol{\Delta}_{nt} + \boldsymbol{\Delta}_{rt} \approx \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)(M_{nt} + M_{rt}) =: \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) M_t \tag{28}$$

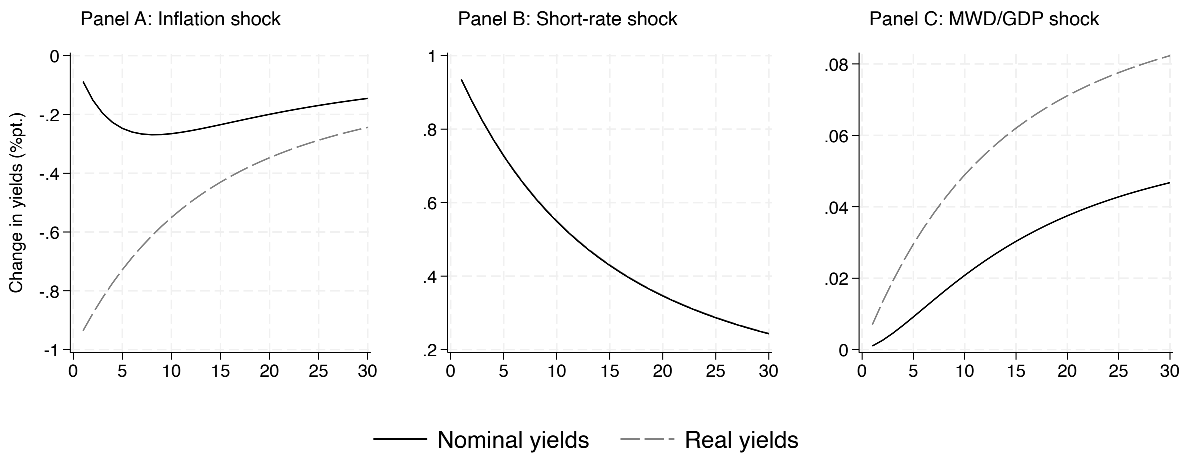

Figure 1:
Cross section of the theoretical response of yields to a unit shock to the factor vector $\mathbf{x}_t := (\pi_t, i_t, m_t)^\top$ at varying maturities.

Note: Plot of (26) and (29) using the calibrated parameters in Table C2.

where $M_t = M_{nt} + M_{rt}$ is the total maturity-weighted-debt. The response of yields to maturity-weighted-debt in our model is characterised in the following proposition.

Proposition 3. Given our approximation, the response of yields, $\mathbf{y}_t^{(\tau)}$ to M_t is

$$\frac{\partial}{\partial M_t}\mathbf{y}_t^{(\tau)} \approx \frac{a}{\tau}\left[\int_0^\tau \boldsymbol{\mathcal{A}}(u)\,\mathrm{d}u\right]^\top \mathbf{\Sigma}\mathbf{\Sigma}^\top \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) \tag{29}$$

Our calibration plotted in Figure 2 shows the response of yields to supply is positive and increasing in maturity. In addition, the real yield curve is affected approximately two times more than the nominal curve by an increase in MWD/GDP at all maturities.

2.8 Forward rates

In Appendix A, we show that instantaneous forward rates in our affine model are given by

$$\tilde{f}_{nt}^{(\tau)} = \mathbf{A}_n'(\tau)^\top \mathbf{q}_t + C_n'(\tau) \tag{30}$$

$$f_{rt}^{(\tau)} = \mathbf{A}_r'(\tau)^\top \mathbf{q}_t + C_r'(\tau) \tag{31}$$

Let $\mathbf{f}_t^{(\tau)} := (\tilde{f}_{nt}^{(\tau)}, f_{rt}^{(\tau)})^\top$ stack the 'nominal' and 'real' forward rate. The following proposition looks at the how forward rates respond to the macroeconomic factors in our model.

Proposition 4. The response of forward rates $\mathbf{f}_t^{(\tau)}$ to:

i. *macroeconomic factors*, $\mathbf{q}_t$ is given by

$$\frac{\partial \mathbf{f}_t^{(\tau)}}{\partial \mathbf{q}_t} = \boldsymbol{\mathcal{A}}'(\tau) \tag{32}$$

ii. the *supply factor*, M_t is given by

$$\frac{\partial \mathbf{f}_t^{(\tau)}}{\partial M_t} = a\boldsymbol{\mathcal{A}}(\tau)^\top \mathbf{\Sigma}\mathbf{\Sigma}^\top \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) \tag{33}$$

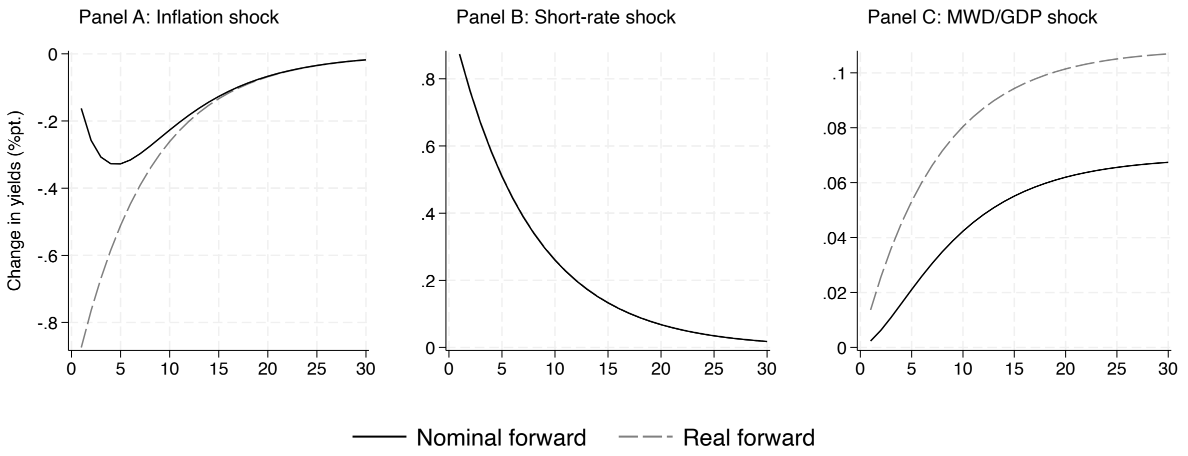

Figure 2:
Cross section of the theoretical response of forward rates to a unit shock to the factor vector $\mathbf{x}_t := (\pi_t, i_t, m_t)^\top$ at varying maturities.

Note: Plot of (32) and (33) using the calibrated parameters in Table C2.

In addition, for

a. *inflation-linked bonds*, the response to

   i. inflation is negative and increasing monotonically with limit zero as $\tau \to \infty$,

   ii. short rates is positive and decreasing monotonically with limit zero as $\tau \to \infty$.

b. *nominal bonds*, if $\kappa_\pi - \kappa_i > 0$, the response to

   i. inflation is negative, and has limit zero as $\tau \to \infty$,

   ii. short rates is positive, and decreasing monotonically with limit zero as $\tau \to \infty$.

The cross-sectional response of instantaneous forward rates is plotted in Figure 2. In our model, forward rates behave similarly to yields in direction, but differ in magnitude. Our calibration shows that the short-end of the forward curve is more sensitive than yields to inflation and interest rate shocks, whereas the long-end of the yield curve is affected more than long-term forward rates. This conforms to standard intuition; yields are the average of future expected short rates, therefore a shock to factors now is averaged out (the τ^{-1} term is present for yields but not forwards). On the other hand, shocks to forward rates decay faster in maturity as futures short rates have a reduced dependence on shocks now. Supply shocks, like yields, have positive effects on the yield curve which are increasing in maturity, and are approximately 20% larger in magnitude than the response by yields.

2.9 Expectations Hypothesis

Proposition 5 (Expectations Hypothesis)**.** A unit shock to the

i. *nominal short rate* raises the expected nominal short rate by same amount as the nominal forward rate, that is

$$\frac{\partial}{\partial i_t} \tilde{f}_{nt}^{(\tau)} = \frac{\partial}{\partial i_t} \mathbb{E}_t[i_{t+\tau}] \tag{34}$$

ii. *real short rate* raises the expected real short rate by same amount as the real forward rate, that is

$$\frac{\partial}{\partial r_t} f_{rt}^{(\tau)} = \frac{\partial}{\partial r_t} \mathbb{E}_t[r_{t+\tau}] \tag{35}$$

Thus we can conclude the Expectations Hypothesis holds for both real and nominal yield curves.

The proof of Proposition 5 is in Appendix A. The Expectations Hypothesis is an artefact of the arbitrageur's optimality condition. With arbitrageurs are present, they transmit short-rate shocks to bond yields, ensuring that yields reflect information about the current and expected future short rates. A significant amount of the empirical finance literature has focused on testing the implications of the Expectations Hypothesis. Campbell and Shiller (1991) proposed two tests: when the yield curve is steeper than usual, both short- and long-term rates are expected to rise. Conversely, if the yield curve is flatter than usual, both short and long-term rates are expected to decline. This approach aligns with studies by Fama and Bliss (1987), Backus et al. (2001), Duffee (2002), and Cochrane and Piazzesi (2005), who regress the excess returns on holding an τ-year bond for an m-year period against the return on an m-year bond held over the same period, in relation to the term structure of interest rates at the start of the period. Bekaert and Hodrick (2001) revisit the two tests introduced by Campbell and Shiller (1991), but adjust the critical values for use in small samples, due to the limitations of data available. Despite these adjustments, they still reject the expectations hypothesis, albeit to a lesser degree than previous studies.

More formally, the risk-neutral evolution of the short rate coincides with the physical measure, and so the Expectations Hypothesis holds (Piazzesi, 2003). In Vayanos and Vila (2021), they investigate a single asset class model but with elastic supply from preferred habitat investors. In this case, arbitrageurs' actions have price impact due to elastic supply, and therefore they bear additional risk when conducting carry trades. The risk-neutral evolution of the short rate diverges from the observed physical measure and so bond yields under-react to a shock to the short rate.

3 Data

3.1 Supply of debt

Our data comes from the Centre for Research in Security Prices (CRSP), which contains issuance details on all the US bonds issued since the 1960s. Although data is available two years from the introduction of TIPS in 1997, we restrict our sample period from January 2001 to exclude initial years when TIPS suffer most from liquidity issues and hence the yields data is less reliable (as documented in D'Amico et al. 2008, Campbell et al. 2009). The CRSP provides data on static characteristics of each bond issued, including but not limited to issue date, maturity date, first coupon date, coupon rate, and face value. In addition to the total face value, our sample is also relatively complete for the face value held by the public - netting out Federal Reserve and inter-agency holdings. We focus our study on the public supply as it better proxies the actual supply available to arbitrageurs. This is of particular importance during our period of observation as the Fed became a substantial player in the Treasuries market due to the expansion of the Fed's balance sheet from quantitative easing.

As in Greenwood and Vayanos (2014), we break each bond's stream of cash-flows into coupon and principle, and treat each part as a zero coupon bond when computing our supply measures. Nominal cash-flows $D_{j,t}^{(\tau)}$ at time t, due τ years from t for bond type $j \in \{n, r\}$ is constructed by

$$D_{jt}^{(\tau)} = \sum_i \text{principal}_{ijt}^{(\tau)} + \sum_i \text{coupon}_{ijt}^{(\tau)}$$

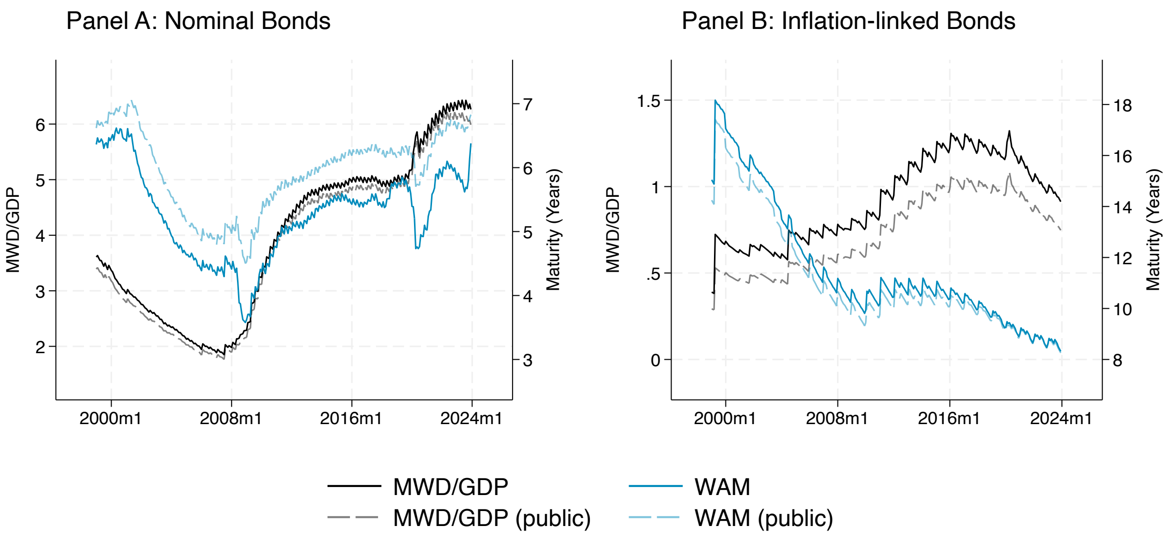

Figure 3:
Bond supply, 1997 to 2024

Note: MWD/GDP is computed with (36). Weighted-Average-Maturity (WAM) is MWD divided by the total value of debt outstanding. 'Public' refers to the supply of debt net Federal holdings of US government bonds.

where principal_{ijt}, coupon_{ijt} is the principal and coupon of bond i of type j at time t. For TIPS bonds, the cash-flows are adjusted by the corresponding inflation compensation factor. We use nominal cashflows as we deflate using nominal GDP as our numeraire.

Motivated by our theoretical predictions, we proxy the factor $\mathbf{\Delta}_t$ with $(-1,1)^\top M_t$. We estimate M_t by its sample counterpart, $m_t := m_{nt} + m_{rt}$, defined by

$$m_{jt} := \sum_{0 \leq \tau \leq T_j} D_{jt}^{(\tau)} \tau \tag{36}$$

and T_j is the maximum maturity for bond type j. Note that we use the face value of debt to compute our supply measures, rather than market prices which would instead give us weighted-average duration. As market prices and yields, our dependent variable, are mechanically related through (4), this creates an endogeneity problem and results in a spurious negative relationship with dollar-duration and yields. For TIPS, we scale the cashflows by the price index, which closely matches the actual payment in nominal terms.

3.2 Yields and macroeconomic factors

We obtain zero-coupon yields and returns for both nominal and inflation-linked bonds from the Federal Reserve, based off Gürkaynak et al. (2006) and Gürkaynak et al. (2008). Nominal bonds are available for one to thirty year maturities, while inflation-indexed bonds are available for one to twenty year maturities. The CPI index, real GDP, VIX index, inflation expectations, the nominal short rate proxied by the 3-month T-Bill rate, and other macroeconomic data are sourced from the Federal Reserve Economic Database (FRED). Our measure of inflation is defined as $\pi_t = \text{CPI}_t/\text{CPI}_{t-12} - 1$, the average of inflation over the last 12 months. We use this over the annualised monthly inflation as it removes seasonality, as well as being the indicator used for adjusting TIPS repayments in financial markets.

Table 1:
Summary statistics, Jan 1999 to Dec 2023

	Mean	Median	SD	Max	Min
Nominal Yields					
2-yr	0.0192	0.0156	0.0154	0.0508	0.00120
5-yr	0.0254	0.0236	0.0132	0.0500	0.00296
10-yr	0.0328	0.0320	0.0127	0.0571	0.00634
20-yr	0.0384	0.0383	0.0127	0.0611	0.0115
30-yr	0.0389	0.0390	0.0111	0.0596	0.0143
Real Yields					
2-yr	0.00155	-0.0000410	0.0144	0.0464	-0.0287
5-yr	0.00651	0.00524	0.0130	0.0347	-0.0185
10-yr	0.0114	0.0104	0.0117	0.0359	-0.0105
20-yr	0.0153	0.0154	0.0101	0.0361	-0.00479
Nominal Instantaneous Forward					
2-yr	0.0233	0.0202	0.0140	0.0497	0.00169
5-yr	0.0349	0.0341	0.0133	0.0621	0.00639
10-yr	0.0436	0.0452	0.0150	0.0680	0.0125
20-yr	0.0424	0.0426	0.0113	0.0631	0.0186
30-yr	0.0375	0.0382	0.00849	0.0540	0.00562
Real Instantaneous Forward					
2-yr	0.00567	0.00488	0.0147	0.0433	-0.0205
5-yr	0.0131	0.0110	0.0128	0.0429	-0.0107
10-yr	0.0186	0.0195	0.0103	0.0375	-0.00296
20-yr	0.0179	0.0176	0.00848	0.0365	0.000690
Macroeconomic Factors					
Nominal short rate	0.0149	0.00937	0.0166	0.0534	0.000114
Inflation	0.0253	0.0216	0.0183	0.0906	-0.0210
1-yr inflation expectations	3.113	3	0.770	5.400	0.400
VIX index	19.91	17.74	8.299	62.67	10.13
GDP	17231562.8	16300760	4651249.8	28514618	10470231
Output gap	-0.000723	0.000693	0.0188	0.0314	-0.112
MWD/GDP					
Nominal bonds	4.016	4.542	1.475	6.431	1.851
Inflation-linked bonds	0.950	0.973	0.232	1.323	0.576
All bonds	4.966	5.584	1.670	7.431	2.603
D/GDP					
Nominal bonds	0.765	0.866	0.238	1.231	0.427
Inflation-linked bonds	0.0901	0.0958	0.0302	0.140	0.0389
All bonds	0.855	0.983	0.266	1.367	0.489
WAM					
Nominal bonds	5.171	5.317	0.624	6.610	3.578
Inflation-linked bonds	11.09	10.65	1.904	16.02	8.320
Aggregate bonds	5.782	5.909	0.556	7.347	4.263
MWD/GDP net of Federal holdings					
Nominal bonds	3.882	4.413	1.460	6.222	1.765
Inflation-linked bonds	0.759	0.791	0.205	1.087	0.435
Aggregate bonds	4.641	5.247	1.637	7.076	2.351

Note: Table summarises the data over the sample period of 2001 to 2023. Yields and forwards data are from Gürkaynak et al. (2006) and Gürkaynak et al. (2008), macroeconomic data is from Federal Reserve Bank of St. Louis (2024), and supply data is constructed with data from Centre for Research in Security Prices (2024). MWD/GDP is maturity-weighted-debt, D/GDP is the debt-to-GDP ratio. the output gap is computed from a Hodrick-Prescott filter of log GDP. Inflation is the 12 month change in the CPI index, $CPI_t/CPI_{t-12} - 1$.

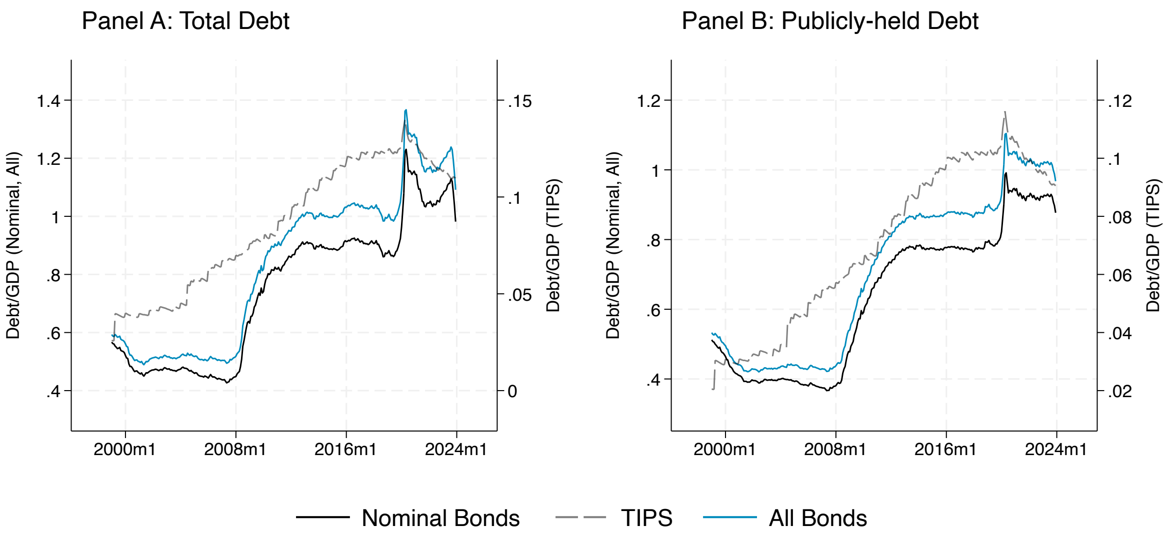

Figure 4:
US Debt-to-GDP, 1997 to 2024

Note: Debt-outstanding is the sum of cashflows of debt outstanding. 'Public' refers to the supply of debt net Federal holdings of US government bonds.

4 Empirical Results

In this section, we evaluate the cross-sectional response of yields to the model factors, and compare our model predictions with the empirical findings using our calibration exercise. We use a two step procedure; first we identify exogenous shocks to the model factors through a recursive vector-autoregression, then compute the response functions by regressing the yields on the shocks and supply.

The identification of exogenous monetary and inflation shocks has long been an important issue in empirical macroeconomics. The classic simultaneity issue - that interest rates rise in response to inflation, while inflation rises in response to lower interest rates - has received significant attention since the 1980s starting with Sims (1980) advocating a more careful approach to identifying causality between macroeconomic variables. A now standard method to overcome these issues is to use a vector-autoregression with a recursive ordering of variables that takes advantage of contemporaneous restrictions on the variables through a Cholesky decomposition (see for example Christiano et al. 1999). Another potential source of bias is that maturity could be endogenous and affected by other variables that influence yields. One possible theoretical explanation is that a government may choose a maturity structure to minimise the expected future interest repayments on its debt. In the data, we note that during recessions the average maturity drops around twenty months as the Treasury issues short-term notes for counter-cyclical fiscal spending.

4.1 Identification

To identify exogenous shocks to model factors, we use a three variable recursive vector-autoregression (VAR) on $\mathbf{x}_t = (\pi_t, i_t, m_t)^\top$. The reduced form VAR takes the form,

$$\mathbf{x}_t = \boldsymbol{\phi}_0 + \sum_{l=1}^{L} \boldsymbol{\Phi}_l \, \mathbf{x}_{t-l} + \mathbf{u}_t \tag{37}$$

where $\boldsymbol{\phi}_0 \in \mathbb{R}^3$, $(\boldsymbol{\Phi}_l)_{l\in\{1,...,L\}} \subset \mathbb{R}^{3\times3}$ are the reduced-form VAR coefficients, and $\mathbf{u}_t \in \mathbb{R}^3$ is the reduced-form VAR error. The exogenous shocks, $e_t \in \mathbb{R}^3$, related to reduced form errors by

$$\mathbf{e}_t = \mathbf{Q}^{-1}\mathbf{u}_t \tag{38}$$

As standard in the literature (Stock and Watson, 2001), we assume $\mathbf{B}$ is lower diagonal, therefore we can identify $\mathbf{e}_t$ via a Cholesky decomposition on the variance of $\mathbf{u}_t$. These restrictions can be interpreted as interest rate shocks having a lagged effect on inflation, and therefore has no contemporaneous impact. We choose the maximum lag length, $L = 14$ that optimises a range of information criterion (AIC, HQIC, LR, FPE)[1] as defined by Lütkepohl (2005), although our results are remain similar with higher lags. Having estimated the orthogonalised shocks $\mathbf{e}_t$, we then regress our choice of dependent variables on the vector of shocks, including aggregated supply and segregated supply as covariates.

4.2 Yield regressions

We first investigate the empirical response of yields to the factors. As before, let $\mathbf{y}_t^{(\tau)} := (\tilde{y}_{nt}^{(\tau)}, y_{rt}^{(\tau)})^\top$ stack 'nominal' and 'real' yields. We estimate the following regressions of yields on exogenous shocks,

$$\mathbf{y}_t^{(\tau)} = \boldsymbol{\alpha}_y^{(\tau)} + \boldsymbol{\beta}_y^{(\tau)}\,\mathbf{e}_t + \boldsymbol{\varepsilon}_{yt}^{(\tau)} \tag{39}$$

where $\boldsymbol{\alpha}_y^{(\tau)} \in \mathbb{R}^2, \boldsymbol{\beta}_y^{(\tau)} \in \mathbb{R}^{2\times3}$. Since yields are influenced by persistent factors beyond just supply and the short rate, such as expected inflation, the residuals from these regressions exhibit serial correlation. We correct for this using Newey and West (1987) standard errors with four lags, following the growth condition $\texttt{lags} = \lfloor \texttt{sample size}^{1/4} \rfloor$ in Greene (2020) where $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is the integer floor function.

The baseline OLS estimates for yields with maturity $\tau = \{1, \ldots, 30\}$ are plotted in Figure 1 and estimates for selected yields are shown in Table 3. We find good support for the model, with the *direction* and *slope* of estimated coefficients corresponding to the theoretical predictions in Section 2. Panel A highlights that nominal yields exhibit a significant positive sensitivity to the nominal short rate i_t, particularly at shorter maturities, with the response declining for longer-term yields ($\beta_i^{(\tau)}$: 1.481 for 2-year vs. 0.340 for 30-year). This is a standard result in the literature, as short-term nominal yields are closely tied to monetary policy decisions through the expectations hypothesis. Our baseline results however indicate nominal yields exhibit little sensitivity to the inflation rate across maturities. In Panel B, similar to nominal yields, real yields are positively sensitive to the short-term interest rate, with this effect diminishing for longer maturities ($\beta_i^{(\tau)}$: 0.690 for 2-year vs. 0.328 for 20-year yields). In addition, real yields exhibit a strong negative response to inflation ($\beta_\pi^{(\tau)}$: -0.702 for 2-year and -0.395 for 5-year yields), consistent with the theoretical expectation that inflation erodes real returns, but over longer horizons, expected to return to $\bar{q}$. This is a credible assumption over most of the sample period chosen as US inflation was near zero and expectations as measured by Michigan survey data showed sufficient central bank credibility. Repeating the analysis excluding the pandemic years does not affect our results. For supply, we observe that point estimates of supply shocks match the positive direction and slope predicted by the model, but are more muted compared with the calibration.

Note that our reported adjusted R^2 statistics are smaller compared with other papers fitting their model, as we focus on the point estimates obtained using only the exogenous shocks that remain after the recursive VAR step. The importance of the first step can be illustrated with a comparison with the naïve OLS estimation of the (37), where $\mathbf{e}_t$ is replaced with the factors themselves, $\mathbf{x}_t$. Although the interest rate factor retains a similar shape, the other factors are all poorly identified. In particular, the

[1]AIC: Akaike's Information Criterion, HQIC: Hannan–Quinn's Information Criterion, LR: Likelihood ratio test of L and $L - 1$, FPE: Akaike's Final Prediction Error

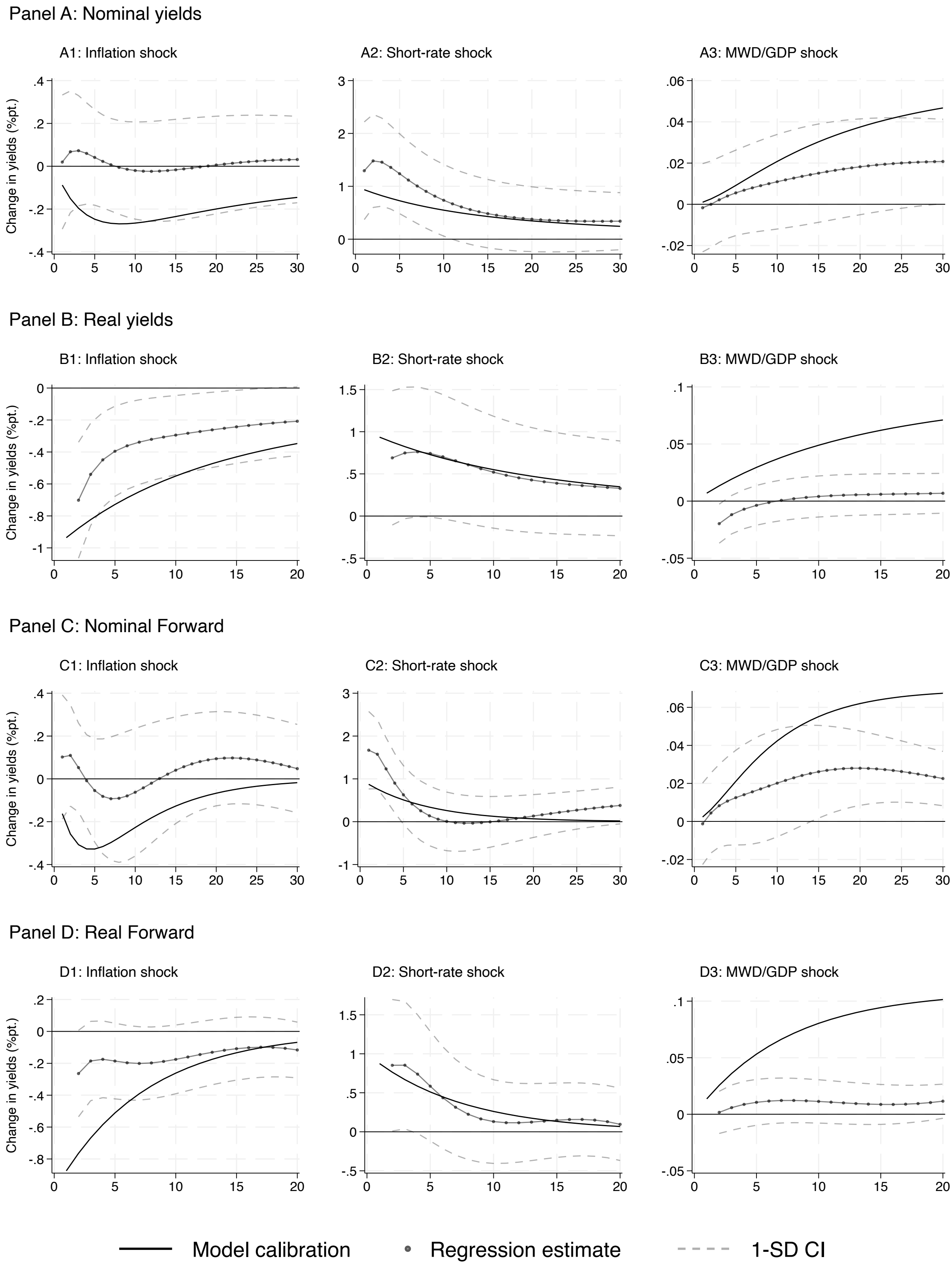

Figure 5:
Cross-section of the theoretical and empirical response of yields and forward rates - baseline specification

Note: Plot of estimated $\boldsymbol{\beta}_f^{(\tau)}$ for $\tau \in (0, T_j]$ from (39) and (40), together with theoretical response in (32),(33).

Table 2:
Cross-section of empirical response of yields to $x_t = (\pi_t, i_t, m_t)$ - baseline specification

	$\beta_\pi^{(\tau)}$	$\beta_i^{(\tau)}$	$\beta_m^{(\tau)}$	$\alpha^{(\tau)}$	N	adj. R^2
Panel A: Nominal yields						
2-yr	0.0676	1.481	0.00000984	0.0207***	286	0.005
	(0.284)	(0.881)	(0.0210)	(0.00219)		
5-yr	0.0413	1.239	0.00556	0.0266***	286	0.005
	(0.223)	(0.755)	(0.0207)	(0.00185)		
10-yr	-0.0201	0.736	0.0110	0.0337***	286	-0.003
	(0.227)	(0.678)	(0.0229)	(0.00175)		
20-yr	0.00574	0.377	0.0182	0.0391***	286	-0.006
	(0.228)	(0.613)	(0.0232)	(0.00171)		
30-yr	0.0314	0.340	0.0208	0.0396***	286	-0.003
	(0.202)	(0.541)	(0.0205)	(0.00149)		
Panel B: Real yields						
2-yr	-0.702	0.690	-0.0198	0.00282	286	0.015
	(0.363)	(0.796)	(0.0171)	(0.00196)		
5-yr	-0.395	0.743	0.00369	0.00767***	286	0.002
	(0.282)	(0.755)	(0.0174)	(0.00182)		
10-yr	-0.294	0.520	0.00407	0.0124***	286	-0.002
	(0.248)	(0.664)	(0.0180)	(0.00163)		
20-yr	-0.208	0.328	0.00688	0.0161***	286	-0.005
	(0.215)	(0.562)	(0.0174)	(0.00141)		

Note: Newey and West (1987) standard errors in parentheses with 4 lags; * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$. Results are shown for OLS estimation of (39), using instantaneous forward rates from Gürkaynak et al. (2006) and Gürkaynak et al. (2008).

Table 3:
Cross-section of empirical response of forwards to $x_t = (\pi_t, i_t, m_t)$ - baseline specification

	$\beta_\pi^{(\tau)}$	$\beta_i^{(\tau)}$	$\beta_m^{(\tau)}$	$\alpha^{(\tau)}$	N	adj. R^2
Panel A: Nominal forwards						
2-yr	0.110	1.575	-0.00444	0.0245***	286	0.011
	(0.237)	(0.807)	(0.0210)	(0.00195)		
5-yr	-0.0544	0.628	-0.0125	0.0357***	286	-0.005
	(0.241)	(0.683)	(0.0249)	(0.00179)		
10-yr	-0.0625	0.00670	-0.0201	0.0442***	286	-0.007
	(0.298)	(0.684)	(0.0282)	(0.00197)		
20-yr	0.0941	0.133	-0.0280	0.0429***	286	0.001
	(0.220)	(0.501)	(0.0195)	(0.00147)		
30-yr	0.0479	0.380	-0.0226	0.0381***	286	0.005
	(0.207)	(0.430)	(0.0143)	(0.00110)		
Panel B: Real forwards						
2-yr	-0.264	0.853	-0.00170	0.00689***	286	-0.002
	(0.271)	(0.842)	(0.0186)	(0.00204)		
5-yr	-0.186	0.585	-0.0106	0.0141***	286	-0.004
	(0.237)	(0.702)	(0.0203)	(0.00177)		
10-yr	-0.175	0.131	-0.0114	0.0193***	286	-0.006
	(0.216)	(0.536)	(0.0192)	(0.00140)		
20-yr	-0.116	0.0964	-0.0116	0.0186***	286	-0.006
	(0.175)	(0.464)	(0.0151)	(0.00117)		

Note: Newey and West (1987) standard errors in parentheses with 4 lags; * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$. Results are shown for OLS estimation of (40), using instantaneous forward rates from Gürkaynak et al. (2006) and Gürkaynak et al. (2008).

effect of inflation for real bonds is roughly a fifth of what we observe with the two step process across the entire curve, which is clearly infeasible as short maturity bond yields should roughly equal the real interest rate, $r_t = i_t - \pi_t$ which has a mechanical one-for-one relationship with inflation. In addition, naive OLS estimates imply the effect of supply is significant and persistently negative, posing a strong challenge to both conventional economic wisdom, as well as the theory developed in this paper.

4.3 Forward rate regressions

As before, let $\mathbf{f}_t^{(\tau)} := (\tilde{f}_{nt}^{(\tau)}, f_{rt}^{(\tau)})^\top$ stack the 'nominal' and 'real' forward rates. We estimate the following regressions of forward rates on exogenous shocks,

$$\mathbf{f}_t^{(\tau)} = \boldsymbol{\alpha}_f^{(\tau)} + \boldsymbol{\beta}_f^{(\tau)} \mathbf{e}_t + \boldsymbol{\varepsilon}_{ft}^{(\tau)} \tag{40}$$

where $\alpha_f^{(\tau)} \in \mathbb{R}^2, \boldsymbol{\beta}_f^{(\tau)} \in \mathbb{R}^{2\times 3}$. The baseline OLS estimates for yields with maturity $\tau = \{1, \ldots, 30\}$ are plotted in Panel C and D of Figure 5, while the estimates for selected yields are shown in Table 3. We find good support for the model with the response direction matching our theoretical predictions. The response to our supply factor is more muted than what our calibration suggests, although the calibration results vary by the sample period selected.

Although for nominal forward rates, the sensitivity to inflation ($\beta_\pi^{(\tau)}$) is small and statistically insignificant across all maturities, the model is able to account for the characteristic hump shape observed in the point estimates, and matches closer when we consider additional control variables discussed in the robustness checks below. The response of nominal and real forward rates to the nominal short rate ($\beta_i^{(\tau)}$) is significant and positive, particularly for shorter maturities. In addition, real forward rates display a negative response to inflation that is consistent with theoretical expectations, with coefficients decreasing in magnitude as maturity increases ($\beta_\pi^{(\tau)}$: -0.264 for 2-year vs. -0.116 for 20-year).

4.4 Robustness

In addition to the regressions above, we perform a number of other robustness tests for the segregated supply case. We first consider the impact of potential confounders; using the same exogenous shocks on $\mathbf{q}_t$, our second step regressions take the general form:

$$\mathbf{y}_t^{(\tau)} = \boldsymbol{\alpha}_y^{(\tau)} + \boldsymbol{\beta}_y^{(\tau)} \mathbf{e}_t + \varphi_y^{(\tau)} X_t + \boldsymbol{\varepsilon}_{yt}^{(\tau)} \tag{41}$$

$$\mathbf{f}_t^{(\tau)} = \boldsymbol{\alpha}_f^{(\tau)} + \boldsymbol{\beta}_f^{(\tau)} \mathbf{e}_t + \varphi_f^{(\tau)} X_t + \boldsymbol{\varepsilon}_{ft}^{(\tau)} \tag{42}$$

where X_t is a control variable. We consider the following control variables with coefficients: stock market volatility $\varphi_{v,f}$ and output gap $\varphi_{x,j}$.

Time-varying risk-aversion

We measure stock market volatility with the implied 30-day volatility on the S&P500 index, or 'VIX' for short. We include the VIX index to capture the effects of time-varying risk-aversion, which in our model affects arbitrageur risk-bearing capacity and the responsiveness of yields to the supply portfolio. Papers such as He and Krishnamurthy (2013) document and model the abnormal behaviour of risk premia during crises. The results are shown in Figure D2. We do not find significant differences with our original baseline specification. As expected, the addition of risk-aversion draws away some of the effect attributed to MWD/GDP shocks, which is where arbitrageur's risk-aversion matters in the model.

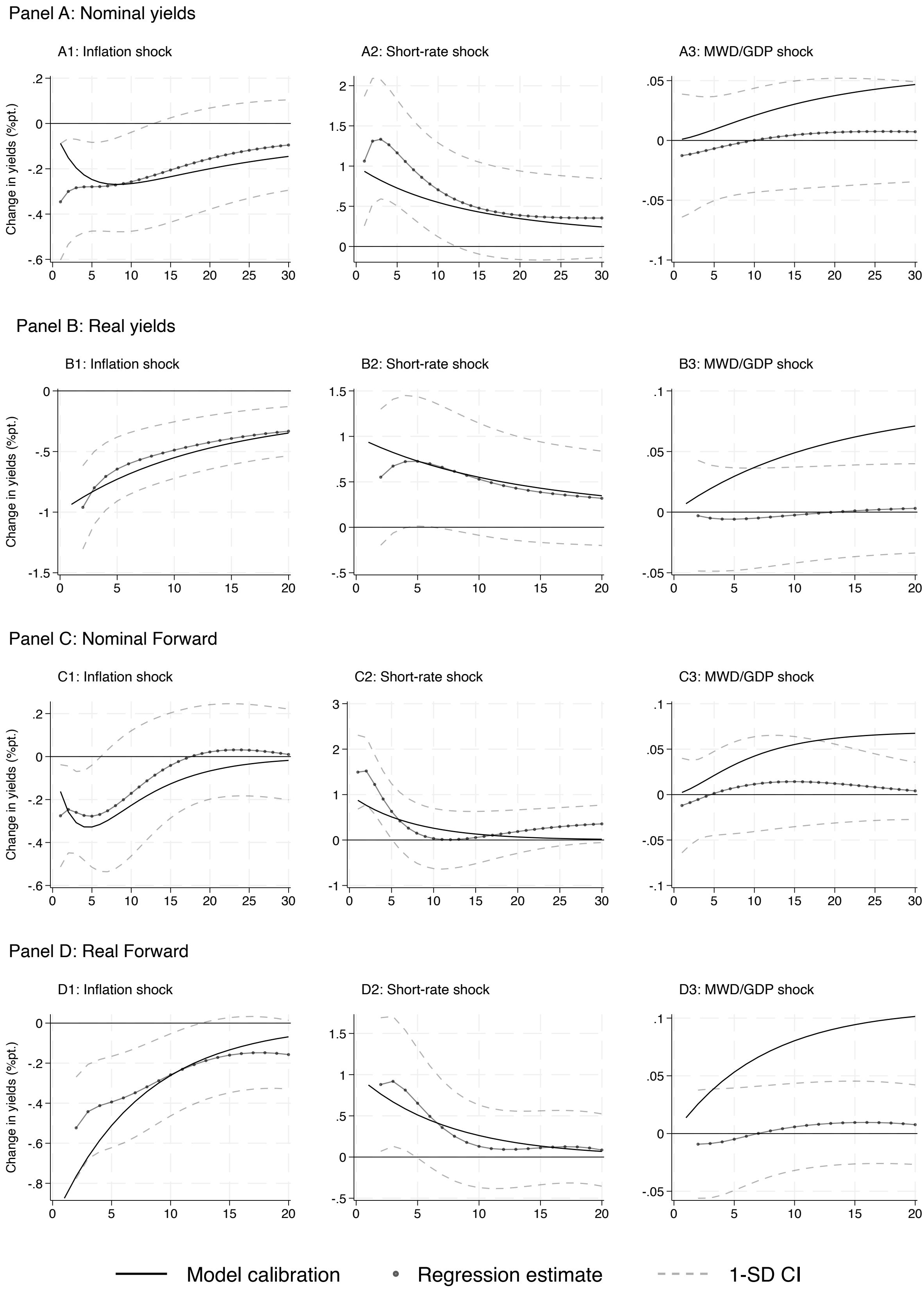

Figure 6:
Cross-section of the theoretical and empirical response of yields and forward rates – controlling for output gap

Note: Plot of estimated $\boldsymbol{\beta}_f^{(\tau)}$ for $\tau \in (0, T_j]$ from (41) and (42), together with their theoretical counterpart in (26), (29), (32), (33). The output gap is measured with a Hodrick-Prescott filter of $\log \text{GDP}$.

Output
The output gap is a fundamental building block in modern macroeconomic models, therefore it is reasonable that it may be a confounding factor in our regression. Macroeconomic models with segmented markets such as in Ray (2019), Ray et al. (2024) and Sims and Wu (2021) provide such a theoretical channel between output and yields. To measure the output gap, we linearly interpolate monthly log real GDP and take the residuals from a Hodrick-Prescott filter with a standard smoothing parameter of 1600. The results of estimating (41) and (42) controlling for the output gap are displayed in Figure 6. We find that point estimates of empirical responses for macroeconomic factors closely match our model calibration for both real and nominal yields, while estimates for supply are almost all positive, but not significant from zero.

As an alternative specification, we also consider an alternative 4-variable VAR with inflation, output, nominal short rate and the MWD. The order follows Stock and Watson (2001), although changing the order of inflation and output does not alter our results. After obtaining the structural shocks, $\mathbf{e}_t$ we proceed as before by regressing yields and forward rates on the shocks. The results are plotted in Figure D3 for yields and forwards. Our results are not systematically changed - there is a slight improvement to the fit for the inflation coefficient, but overall there is no improvement in model fit as measured by adjusted R^2, suggesting that the output gap does not add any additional explanatory power over the existing variables used.

5 Conclusion

In this paper, we jointly model the real and nominal term structure in the modern preferred habitat framework of Vayanos and Vila (2021), and explore the connections between the two curves beyond the traditional link, break-even inflation. Consistent with other preferred habitat models, our model predicts that an increase in supply for one particular maturity should raise bond yields across the yield curve, termed 'global effects'. In addition, we show that the joint supply is in fact a determining factor for term-premia in either yield curve, as arbitrageurs in our model consider the total amount of debt available. Furthermore, we provide a derivation for the case when supply is influenced by an exogenous stochastic factor. In addition to supply, we form explicit relationships for how macroeconomic factors - inflation and interest rates should affect each yield curve. We show analytically the sign and slope properties of the yield response functions to our model factors, and calibrate the model using US data from 2001 to 2023.

To evaluate the effectiveness of the model, we test the theoretical predictions using US Treasury and TIPS yield data, and compare the empirically estimated response functions to the calibrated model. By identifying exogenous shocks to the short rate and inflation through a Cholesky ordering of variables in a recursive vector-autoregression, we regress the shocks on yields and forward rates to find good support for the model's predictions. In particular, a shock to maturity-weighted-debt-to-GDP by one unit increases 20-yr nominal yields by around 305bps, and 20-yr real yields by 116bps. We consider a range of specifications and robustness tests and show our results are not significantly unaltered by these changes.

Appendix A Proof of Theoretical Results

A.1 Arbitrageur optimality

Proof of Theorem 1. Substituting real returns from (12), we can rewrite the arbitrageur's budget constraint in (9) as

$$\mathrm{d}W_t = \left[W_t r_t + \sum_{j=n,r} \int_0^T x_{jt}^{(\tau)} \left(\mu_{jt}^{(\tau)} - r_t\right) \mathrm{d}\tau\right] \mathrm{d}t - \left[\sum_{j=n,r} \int_0^T x_{jt}^{(\tau)} \mathbf{A}_j(\tau)\, \mathrm{d}\tau\right]^\top \mathbf{\Sigma}\, \mathrm{d}\mathbf{B}_t$$

and thus the objective function (8) as

$$\sum_{j=n,r} \int_0^T x_{jt}^{(\tau)} (\mu_{jt}^{(\tau)} - r_t)\, \mathrm{d}\tau - \frac{a}{2} \left[\sum_{j=n,r} \int_0^T x_{jt}^{(\tau)} \mathbf{A}_j(\tau)\, \mathrm{d}\tau\right]^\top \mathbf{\Sigma}\mathbf{\Sigma}^\top \left[\sum_{j=n,r} \int_0^T x_{jt}^{(\tau)} \mathbf{A}_j(\tau)\, \mathrm{d}\tau\right] \tag{A.1}$$

Point-wise maximisation of (A.1) yields the arbitrageur's optimality conditions in (14). □

A.2 Affine equilibrium solution

Proof of Theorem 2. To solve the ODE system in (18), we diagonalise $\mathbf{\Gamma}^\top$ by

$$\mathbf{\Gamma}^\top = \mathbf{P}^{-1} \mathrm{diag}(\nu_1, \nu_2) \mathbf{P} \tag{A.2}$$

where $\mathbf{P}$ is an ordered matrix of eigenvectors, with inverse $\mathbf{P}^{-1}$,

$$\mathbf{P} := \begin{pmatrix} \nu_1 - \kappa_i & \nu_2 - \kappa_i \\ \gamma & \gamma \end{pmatrix}, \qquad \mathbf{P}^{-1} = \frac{1}{\gamma(\nu_1 - \nu_2)} \begin{pmatrix} \gamma & \kappa_i - \nu_2 \\ -\gamma & \nu_1 - \kappa_i \end{pmatrix}$$

Following Vayanos and Vila (2021), we pre-multiply the system by $\mathbf{P}$, to get:

$$\mathbf{P}\mathbf{A}_j'(\tau) + \mathrm{diag}(\nu_1, \nu_2)\mathbf{P}\mathbf{A}_j(\tau) - \mathbf{P}\mathbf{c}_j = \mathbf{0}$$

Integrating with the terminal condition, we obtain

$$\mathbf{P}\mathbf{A}_j(\tau) = \mathrm{diag}\left(\frac{1 - e^{-\nu_1 \tau}}{\nu_1}, \frac{1 - e^{-\nu_2 \tau}}{\nu_2}\right) \mathbf{P}\mathbf{c}_j$$

Now using the fact:

$$\mathrm{diag}\left(\frac{1 - e^{-\nu_1 \tau}}{\nu_1}, \frac{1 - e^{-\nu_2 \tau}}{\nu_2}\right) = \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \mathbf{I}_2 + \mathrm{diag}\left(0, \ \frac{1 - e^{-\nu_2 \tau}}{\nu_2} - \frac{1 - e^{-\nu_1 \tau}}{\nu_1}\right)$$

we can write the solution as:

$$\mathbf{A}_j(\tau) = \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \mathbf{c}_j + \mathbf{P}^{-1} \mathrm{diag}\left(0, \ \frac{1 - e^{-\nu_2 \tau}}{\nu_2} - \frac{1 - e^{-\nu_1 \tau}}{\nu_1}\right) \mathbf{P}\mathbf{c}_j$$

Multiplying out the system yields the solution of (18). For (19), the proof of $C_j(\tau)$ is provided in text. □

A.3 Factor responses

Lemma 3. For $\tau \in (0, T]$, the function, $g(\tau; \nu) := \frac{1 - e^{-\nu \tau}}{\nu}$ satisfies the following properties:

i. $g(\tau;\nu) > 0$,

ii. $\frac{\partial}{\partial \tau} g(\tau;\nu) > 0$,

iii. $\frac{\partial}{\partial \nu} g(\tau;\nu) < 0$, and thus $g(\tau,\nu_2) < g(\tau;\nu_1)$ for all $\tau \in (0,T]$,

iv. $\frac{\partial^2}{\partial \nu\, \partial \tau} g(\tau;\nu) < 0$

v. $\frac{\partial^2}{\partial \tau^2} g(\tau;\nu) < 0$,

vi. $\frac{\partial^3}{\partial \tau^2\, \partial \nu} g(\tau;\nu) < 0$

vii. $\frac{g(\tau;\nu)}{\tau} > 0$,

viii. $\frac{\partial}{\partial \tau} \frac{g(\tau;\nu)}{\tau} \leq 0$,

ix. $\lim_{\tau\to\infty} \frac{g(\tau;\nu)}{\tau} = 0$ and $\lim_{\tau\to 0} \frac{g(\tau;\nu)}{\tau} = 1$

Proof. For

i. if $\tau \in (0,T]$, then $e^{-\nu\tau} < 1 \iff 1 - e^{-\nu\tau} > 0 \iff g(\tau) > 0$

ii. the derivative is $g'(\tau) = e^{-\nu\tau}$ which satisfies $1 > g'(\tau) > 0$ for $\tau \in (0,T]$.

iii. the derivative with respect to the parameter, ν is

$$\frac{\partial}{\partial \nu} g(\tau;\nu) = -\frac{\tau e^{-\nu\tau}}{\nu} \tag{A.3}$$

which is clearly negative. The second result is by the Mean Value Theorem.

iv. Note

$$\frac{\partial^2}{\partial \nu\, \partial \tau} g(\tau;\nu) = -\tau e^{-\nu\tau} < 0 \tag{A.4}$$

thus $g'(\tau;\nu_2) < g'(\tau;\nu_1)$.

v. We have

$$\frac{\partial^2}{\partial \tau^2} g(\tau;\nu) = -\nu e^{-\nu\tau} < 0 \tag{A.5}$$

vi. We have

$$\frac{\partial^3}{\partial \tau^2\, \partial \nu} g(\tau;\nu) = (\nu\tau - 1)\, e^{-\nu\tau} \geq (\nu\tau - 1)(\nu\tau + 1) = \nu^2\tau^2 - 1 \tag{A.6}$$

Using the inequality $1 + \nu\tau \leq e^{\nu\tau}$,

vii. for $\tau \in (0,T]$, as $g(\tau) > 0$, it clearly holds that $g(\tau)/\tau > 0$.

viii. the derivative is

$$\frac{\partial}{\partial \tau} \frac{g(\tau;\nu)}{\tau} = \frac{\nu e^{-\nu\tau}\tau - 1 + e^{-\nu\tau}}{\nu\tau^2} = \frac{e^{-\nu\tau}(1+\nu\tau) - 1}{\nu\tau^2} \leq 0 \tag{A.7}$$

from the inequality $(1+x)^{-1} \geq e^{-x}$.

ix. To show the limit, note $\lim_{\tau\to\infty} g(\tau) = 1$ and so $\lim_{\tau\to\infty} g(\tau)/\tau = 0$. The case at zero is trivial.

□

Proof of Proposition 1. For

- $j = r$, Lemma 3 provides the result.
- $j = n$, and
  (a) $k = \pi$, we have $A_{\pi r}(\tau) = \xi(g_2(\tau) - g_1(\tau))$. For $\tau \in (0, T]$ we have that $g_2 < g_1$ from Lemma 3. Note the sign of ξ depends on $\nu_2 - \kappa_i$ as $\nu_2 > \nu_1$ by assumption; we look at $\nu_2 - \kappa_i$,

$$2(\nu_2 - \kappa_i) = (\kappa_\pi - \kappa_i) + \sqrt{(\kappa_\pi - \kappa_i)^2 - 4\phi\gamma} \tag{A.8}$$

  which is positive (and > 1) if and only if $\kappa_\pi - \kappa_i > 0$, and negative (and < -1) if and only if $\kappa_\pi - \kappa_i < 0$.

  (b) $k = i$, we have $A_{in}(\tau) = \xi(g_1(\tau) - g_2(\tau)) + g_2(\tau)$.
    i. If $\kappa_\pi - \kappa_i > 0$, we have $\xi(g_1 - g_2) > g_1 - g_2$, then for $\tau \in (0, T]$,

$$A_{in}(\tau) > g_1(\tau) > 0 \tag{A.9}$$

    ii. If $\kappa_\pi - \kappa_i < 0$, we have

$$A_{in}(\tau) = \xi g_1 + (|\xi| - 1)g_2 \tag{A.10}$$

    and so $\xi(g_1 - g_2) = -|\xi|(g_1 - g_2)$, thus

$$A_{in} > 0 \iff |\xi|g_2 - |\xi|g_1 + g_2 > 0 \iff g_1 < \frac{|\xi| + 1}{|\xi|} g_2 \tag{A.11}$$

    and

$$g_1 < \left(1 + \frac{1}{|\xi|}\right) g_2 \iff g_1 - g_2 < \frac{1}{|\xi|} g_2 \tag{A.12}$$

□

A.4 Forward rates

Definition 4 (Forward rate)**.** The nominal forward rate for bond type $j \in \{n, r\}$ is defined as

$$\tilde{f}_{jt}^{(\tau-\delta,\tau)} = -\frac{1}{\delta}\left(\log \tilde{P}_{jt}^{(\tau)} - \log \tilde{P}_{jt}^{(\tau-\delta)}\right) \tag{A.13}$$

and the real forward rate is

$$f_{jt}^{(\tau-\delta,\tau)} = -\frac{1}{\delta}\left(\log P_{jt}^{(\tau)} - \log P_{jt}^{(\tau-\delta)}\right) \tag{A.14}$$

Lemma 5 (Instantaneous forward rates)**.** The instantaneous real forward rate, $f_{jt}^{(\tau)}$ is the limit as $\delta \to 0$ of the $(\tau - \delta, \tau)$ forward rate

$$f_{jt}^{(\tau)} := \lim_{\delta \to 0} f_{jt}^{(\tau-\delta,\tau)} \tag{A.15}$$

and analogously for nominal forward rates. It holds that

$$f_{jt}^{(\tau)} = -\frac{\partial P_{jt}^{(\tau)}}{\partial \tau} \tag{A.16}$$

Proof. Viewing prices as a function of maturity, the instantaneous forward is just the definition of the derivative. □

Proof of Proposition 4. For

i. $j = r$, the result follows from Lemma 3.

ii. $j = n$, and

(a) $k = \pi$, we have

$$A'_{\pi n}(\tau) = \xi(g'_2(\tau) - g'_1(\tau)) \tag{A.17}$$

which is negative if and only if $\kappa_\pi - \kappa_i > 0$ through ξ.

(b) $k = i$, we have

$$A'_{in}(\tau) = \xi(g'_1(\tau) - g'_2(\tau)) + g'_2(\tau) \tag{A.18}$$

If $\kappa_\pi - \kappa_i > 0$, this is clearly positive from Lemma 3. If $\kappa_\pi - \kappa_i < 0$, then an analogous condition to (A.12) can be derived.

□

A.5 Expectations Hypothesis

Proof. The solution to the multi-variate Ornstein-Uhlenbeck process is

$$\mathbf{q}_{t+\tau} = \bar{\mathbf{q}}\left(\mathbf{I}_2 - e^{-\mathbf{\Gamma}\tau}\right) + e^{-\mathbf{\Gamma}\tau}\mathbf{q}_t + \boldsymbol{\varepsilon}_t^{(\tau)}; \quad \boldsymbol{\varepsilon}_t^{(\tau)} = \int_t^{t+\tau} e^{-\mathbf{\Gamma}(t+\tau-s)}\mathbf{\Sigma}\,\mathrm{d}\mathbf{B}_s \tag{A.19}$$

Taking conditional expectations, we find

$$\mathbb{E}_t[\mathbf{q}_{t+\tau}] = (\mathbf{I}_3 - e^{-\mathbf{\Gamma}\tau})\bar{\mathbf{q}} + e^{-\mathbf{\Gamma}\tau}\mathbf{q}_t \tag{A.20}$$

Thus the change in expectations of $\mathbf{q}_{t+\tau}$ is the Jacobian

$$\frac{\partial}{\partial \mathbf{q}_t}\,\mathbb{E}_t[\mathbf{q}_{t+\tau}] = e^{-\mathbf{\Gamma}^\top\tau} = \mathbf{P}^{-1}\mathrm{diag}(e^{-\nu_1\tau}, e^{-\nu_2\tau})\mathbf{P} \tag{A.21}$$

where the final equality uses the same diagonalisation as in Section A.2; that is $\mathbf{\Gamma}^\top = \mathbf{P}^{-1}\mathrm{diag}(\nu_1, \nu_2)\mathbf{P}$.

Nominal expectations hypothesis
To obtain the effect of the nominal short rate on expected nominal short rates, we require the $[2, 2]$ element, i.e.

$$\frac{\partial}{\partial i_t}\,\mathbb{E}_t[i_{t+\tau}] = \mathbf{e}_2^\top\mathbf{P}^{-1}\mathrm{diag}(e^{-\nu_1\tau}, e^{-\nu_2\tau})\mathbf{P}\mathbf{e}_2 \tag{A.22}$$

For the nominal forward rate, the response is given by

$$\frac{\partial \tilde{f}_{nt}^{(\tau)}}{\partial \mathbf{q}_t} = \mathbf{A}'_n(\tau) = \mathbf{P}^{-1}\mathrm{diag}(e^{-\nu_1\tau}, e^{-\nu_2\tau})\mathbf{P}\mathbf{c}_n \tag{A.23}$$

where the second equality uses the solution in the form of (A.3) in the Appendix. The response to the nominal short rate is the second component,

$$\frac{\partial \tilde{f}_{nt}^{(\tau)}}{\partial i_t} = \mathbf{e}_2^\top \mathbf{P}^{-1} \mathrm{diag}(e^{-\nu_1\tau}, e^{-\nu_2\tau}) \mathbf{P} \mathbf{c}_n \tag{A.24}$$

Noting that $\mathbf{c}_n \equiv \mathbf{e}_2$, it is clear that to see that (A.22) is identical to (A.24).

Real expectations hypothesis

For the expected real interest rate, we first look at

$$\frac{\partial}{\partial \mathbf{q}_t} \mathbb{E}_t[r_{t+\tau}] = \frac{\partial}{\partial \mathbf{q}_t} \mathbb{E}_t[\mathbf{q}_{t+\tau}] \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix} \right) \tag{A.25}$$

$$= \mathbf{P}^{-1} \mathrm{diag}(e^{-\nu_1\tau}, e^{-\nu_2\tau}) \mathbf{P} \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix} \right) \tag{A.26}$$

On the other hand, the real forward has response

$$\frac{\partial f_{rt}^{(\tau)}}{\partial \mathbf{q}_t} = \mathbf{A}_r'(\tau) = \mathbf{P}^{-1} \mathrm{diag}(e^{-\nu_1\tau}, e^{-\nu_2\tau}) \mathbf{P} \mathbf{c}_r \tag{A.27}$$

where $\mathbf{A}_r'(\tau)$ is again taken from (A.3). Since $\mathbf{c}_r = (-1, 1)^\top$, (A.27) is identical to (A.26). □

Appendix B Theoretical Extensions

B.1 Stochastic supply

We assume there is an additional stochastic supply factor that affects the supply across both yield curves to a varying degree. Define this factor as β_t, which follows the Ornstein-Uhlenbeck process

$$d\beta_t = -\kappa_\beta \, \beta_t \, dt + \sigma_\beta \, dB_{\beta,t} \tag{B.1}$$

Supply for both curves for any maturity τ is an affine function of this supply factor

$$s_{jt}^{(\tau)} = \zeta_j(\tau) + \theta_j(\tau) \, \beta_t \tag{B.2}$$

To accommodate this additional state variable, we define $\mathbf{x}_t := (\pi_t, i_t, \beta_t)^\top$ as following the multivariate Ornstein-Uhlenbeck process

$$d\mathbf{x}_t = \mathcal{K}(\bar{\mathbf{x}} - \mathbf{x}_t) \, dt + \mathcal{S} \, d\hat{\mathbf{B}}_t \tag{B.3}$$

with parameter matrices

$$\mathcal{K} := \begin{pmatrix} \kappa_\pi & \gamma & 0 \\ -\phi & \kappa_i & 0 \\ 0 & 0 & \kappa_\beta \end{pmatrix} \qquad \mathcal{S} := \begin{pmatrix} \sigma_\pi & \sigma_{\pi i} & 0 \\ \sigma_{i\pi} & \sigma_\pi & 0 \\ 0 & 0 & \sigma_\beta \end{pmatrix} \tag{B.4}$$

and $\hat{\mathbf{B}}_t := (B_{\pi t}, B_{it}, B_{\beta t})^\top$ is a vector of standard independent Brownian motion terms. The rest of the setup is the same as before; we conjecture bond prices are (exponentially) affine in the state vector $\mathbf{x}_t$. To solve this model, we begin from the arbitrageur's optimality constraint and substitute market clearing, $x_{jt}^{(\tau)} = s_{jt}^{(\tau)}$ to get

$$\begin{aligned} \mathbf{A}_j'(\tau)^\top \mathbf{x}_t + C_j'(\tau) + \mathbf{A}_j(\tau)^\top \mathcal{K}(\mathbf{x}_t - \bar{\mathbf{x}}) + \frac{1}{2} \mathbf{A}_j(\tau)^\top \mathcal{S}\mathcal{S}^\top \mathbf{A}_j(\tau) - i_t + \pi_t \, \mathbf{1}_{\{j=r\}} \\ = a \mathbf{A}_j(\tau)^\top \mathcal{S}\mathcal{S}^\top \int_0^T \sum_{j=n,r} \left[\zeta_j(\tau) + \theta_j(\tau) \, \beta_t\right] \mathbf{A}_j(\tau) \, d\tau \end{aligned} \tag{B.5}$$

Identifying linear terms in $\mathbf{x}_t$ yields the system of ODEs

$$\mathbf{A}_j'(\tau) + \mathbf{M}^\top \mathbf{A}_j(\tau) - \mathbf{a}_j = \mathbf{0} \tag{B.6}$$

where the coefficient matrix $\mathbf{M}$ is defined as

$$\mathbf{M} := \mathcal{K} - a \mathcal{S}\mathcal{S}^\top \left[\int_0^T \sum_{j=n,r} \theta_j(\tau) \, \mathbf{A}_j(\tau) \, d\tau \right] \mathbf{e}_3^\top \tag{B.7}$$

and $\mathbf{a}_j := (-1, \mathbf{1}_{\{j=r\}}, 0)^\top$, $\mathbf{e}_3 := (0, 0, 1)^\top$ are constant vectors.

The rest of the solution essentially follows the same method in Appendix A. Let $\nu_1, \ldots, \nu_3$ be the eigenvalues of $\mathbf{M}^\top$, and $\mathbf{\Lambda}$ be a matrix of corresponding eigenvectors of $\mathbf{M}^\top$. As we showed before, the solution is in general obtained as:

$$\mathbf{A}_j(\tau) = \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \mathbf{a}_j + \mathbf{\Lambda}^{-1} \text{diag} \left(0, \frac{1 - e^{-\nu_2 \tau}}{\nu_2} - \frac{1 - e^{-\nu_1 \tau}}{\nu_1}, \frac{1 - e^{-\nu_3 \tau}}{\nu_3} - \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \right) \mathbf{\Lambda} \mathbf{a}_j$$

Note that ν_1, ν_2 are identical to the deterministic supply case, while

$$\nu_3 = \kappa_\beta - a\sigma_\beta^2 \sum_{j=n,r} \int_0^T \theta_j(\tau)\, A_{\beta j}(\tau)\, \mathrm{d}\tau \tag{B.8}$$

satisfies the fixed point condition, which follows from the assumed structure of the $\mathbf{S}$ matrix.

Theorem 6. The solution for the first two factors, $\mathbf{A}_{\mathbf{q}j}(\tau) := (A_{\pi j}(\tau), A_{ij}(\tau))^\top$ are identical to the deterministic supply case.

$$\mathbf{A}_{\mathbf{q}j}(\tau) = \frac{1-e^{-\nu_1\tau}}{\nu_1}\mathbf{c}_j + \mathbf{P}^{-1}\mathrm{diag}\left(0, \frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right)\mathbf{P}\mathbf{c}_j$$

The supply coefficient then solves the ODE

$$A'_{\beta j}(\tau) + \nu_3\, A_{\beta j}(\tau) = a\mathbf{I}_\mathbf{q}^\top \mathbf{\Sigma}\mathbf{\Sigma}^\top \mathbf{A}_{\mathbf{q}j}(\tau); \qquad A_{\beta j}(0) = 0 \tag{B.9}$$

where $\mathbf{I}_\mathbf{q} := \sum_{j=n,r}\int_0^T \theta_j(\tau)\mathbf{A}_{\mathbf{q}j}(\tau)\,\mathrm{d}\tau$ is derived from the solution to the first two factors. The solution for $A_{\beta j}$ satisfies

$$A_{\beta j}(\tau) = e^{-\nu_3\tau} a\mathbf{I}_\mathbf{q}^\top \mathbf{\Sigma}\mathbf{\Sigma}^\top \int_0^\tau e^{\nu_3 u}\mathbf{A}_{\mathbf{q}j}(u)\,\mathrm{d}u \tag{B.10}$$

where ν_3 solves the fixed point condition

$$\nu_3 = \kappa_\beta - a^2\sigma_\beta^2\, \mathbf{I}_\mathbf{q}^\top \mathbf{\Sigma}\mathbf{\Sigma}^\top \sum_{j=n,r}\int_0^T \theta_j(\tau)\, e^{-\nu_3\tau}\int_0^\tau e^{\nu_3 u}\mathbf{A}_{\mathbf{q}j}(u)\,\mathrm{d}u\,\mathrm{d}\tau \tag{B.11}$$

Proof. We first look at $\mathbf{M}$:

$$\mathbf{M}^\top \mathbf{A}_j(\tau) = \begin{pmatrix} \mathbf{\Gamma}^\top \mathbf{A}_{\mathbf{q}j}(\tau) \\ \kappa_\beta A_{\beta j}(\tau) - a\sigma_\beta^2 I_\beta - a\mathbf{I}_\mathbf{q}^\top \mathbf{\Sigma}\mathbf{\Sigma}^\top \mathbf{A}_{\mathbf{q}j} \end{pmatrix} \tag{B.12}$$

where $I_\beta := \sum_{j=n,r}\int_0^T \theta_j(\tau)A_{\beta j}(\tau)\,\mathrm{d}\tau$. The ODEs are

$$\mathbf{A}'_{\mathbf{q}j}(\tau) + \mathbf{\Gamma}^\top \mathbf{A}_{\mathbf{q}j}(\tau) - \mathbf{c}_j = 0 \tag{B.13}$$

$$A'_{\beta j}(\tau) + \nu_3 A_{\beta j}(\tau) = \mathbf{I}_\mathbf{q}^\top \mathbf{\Sigma}\mathbf{\Sigma}^\top \mathbf{A}_{\mathbf{q}j}(\tau) \tag{B.14}$$

Integrate from zero to τ using an integrating factor of $e^{\nu_3\tau}$ to yield the result. □

As presented in the fixed supply case, an equilibrium exists when $a = 0$. For $a > 0$, Ray et al. (2024) show that in a neighbourhood around $a \approx 0$, an equilibrium exists.

Appendix C Calibration

To calibrate the model, we use an Euler-Maruyama scheme to discretise the Ornstein-Uhlenbeck process with its discrete time analogue, the VAR(1) process. With time step Δ, we discretise (2) as follows:

$$\begin{aligned} \mathbf{q}_{t+\Delta} - \mathbf{q}_t &= -\boldsymbol{\Gamma}(\mathbf{q}_t - \bar{\mathbf{q}})\Delta + \boldsymbol{\Sigma}(\mathbf{B}_{t+\Delta} - \mathbf{B}_t) \\ \iff \mathbf{q}_{t+\Delta} &= \boldsymbol{\Gamma}\Delta\bar{\mathbf{q}} + (\mathbf{I}_2 - \boldsymbol{\Gamma}\Delta)\mathbf{q}_t + \boldsymbol{\Sigma}(\mathbf{B}_{t+\Delta} - \mathbf{B}_t) \end{aligned} \tag{C.1}$$

and perform the vector auto-regression

$$\mathbf{q}_{t+\Delta} = \mathbf{a} + \mathbf{C}\mathbf{q}_t + \mathbf{u}_{t+\Delta} \tag{C.2}$$

where $\Delta = \frac{1}{12}$ as our data is monthly. Comparing (C.1) with (C.2), we obtain an estimate $\hat{\boldsymbol{\Gamma}}$ of $\boldsymbol{\Gamma}$ from the regression estimates, $\hat{\mathbf{C}}$ through

$$\hat{\boldsymbol{\Gamma}} = \Delta^{-1}(\mathbf{I}_2 - \hat{\mathbf{C}}) \tag{C.3}$$

Furthermore, denoting the variance-covariance matrix of $\mathbf{u}_{t+\Delta}$ by $\mathbf{V} := \mathrm{var}(\mathbf{u}_{t+\Delta})$, we have

$$\widehat{\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top} = \Delta^{-1}\hat{\mathbf{V}} \tag{C.4}$$

The estimates of the VAR in (C.2), $\hat{\mathbf{C}}$ and $\hat{\mathbf{V}}$ are presented in Table C1 below, and the estimated parameters in Table C2.

Table C1:
Results of vector auto-regression (C.2)

Slope estimates, $\hat{\mathbf{C}}$	*Inflation,* π_t	*Short rate,* i_t
Lagged inflation, $\pi_{t-\Delta}$	0.972***	-0.000440
	(0.0222)	(0.01744)
Lagged nominal short rate, $i_{t-\Delta}$	-.00406	0.989***
	(0.00872)	(0.00685)
Constant, $\mathbf{a}$	0.000824	0.0000716
	(0.000542)	(0.000213)
N	254	254
Estimated covariance matrix, $\hat{\mathbf{V}}$	$u_{\pi,t}$	$u_{i,t}$
$u_{\pi,t}$	0.215	0.0150
$u_{i,t}$	0.0150	0.0304

Note: Large sample standard errors in parentheses; * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$. Maximum likelihood estimation of equation (C.2). Sub-sample period used is Jan 2001 to March 2022.

Table C2:
Estimated parameters of the multivariate Ornstein-Uhlenbeck process in (2)

κ_π	γ	ϕ	κ_i
0.339	0.00527	0.04867	0.135

Appendix D Empirical Appendix

D.1 Public supply

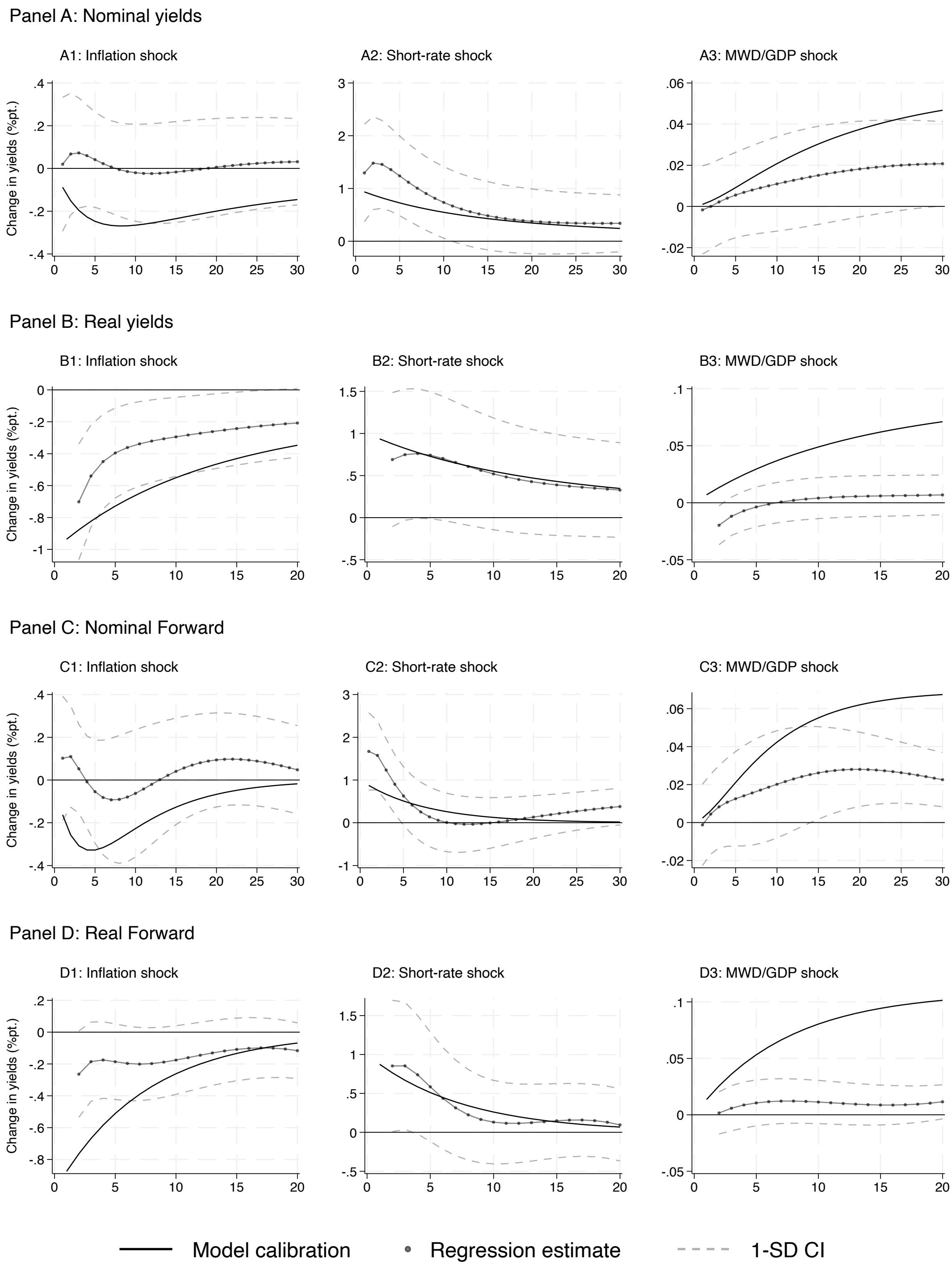

Figure D1:
Cross-section of the theoretical and empirical response of yields and forward rates – with public supply

Note: Plot of estimated $\boldsymbol{\beta}_f^{(\tau)}$ for $\tau \in (0, T_j]$ from (41) and (42) using public supply available (net of Federal holdings), together with their theoretical counterpart in (26), (29), (32), (33)

D.2 Controlling for VIX

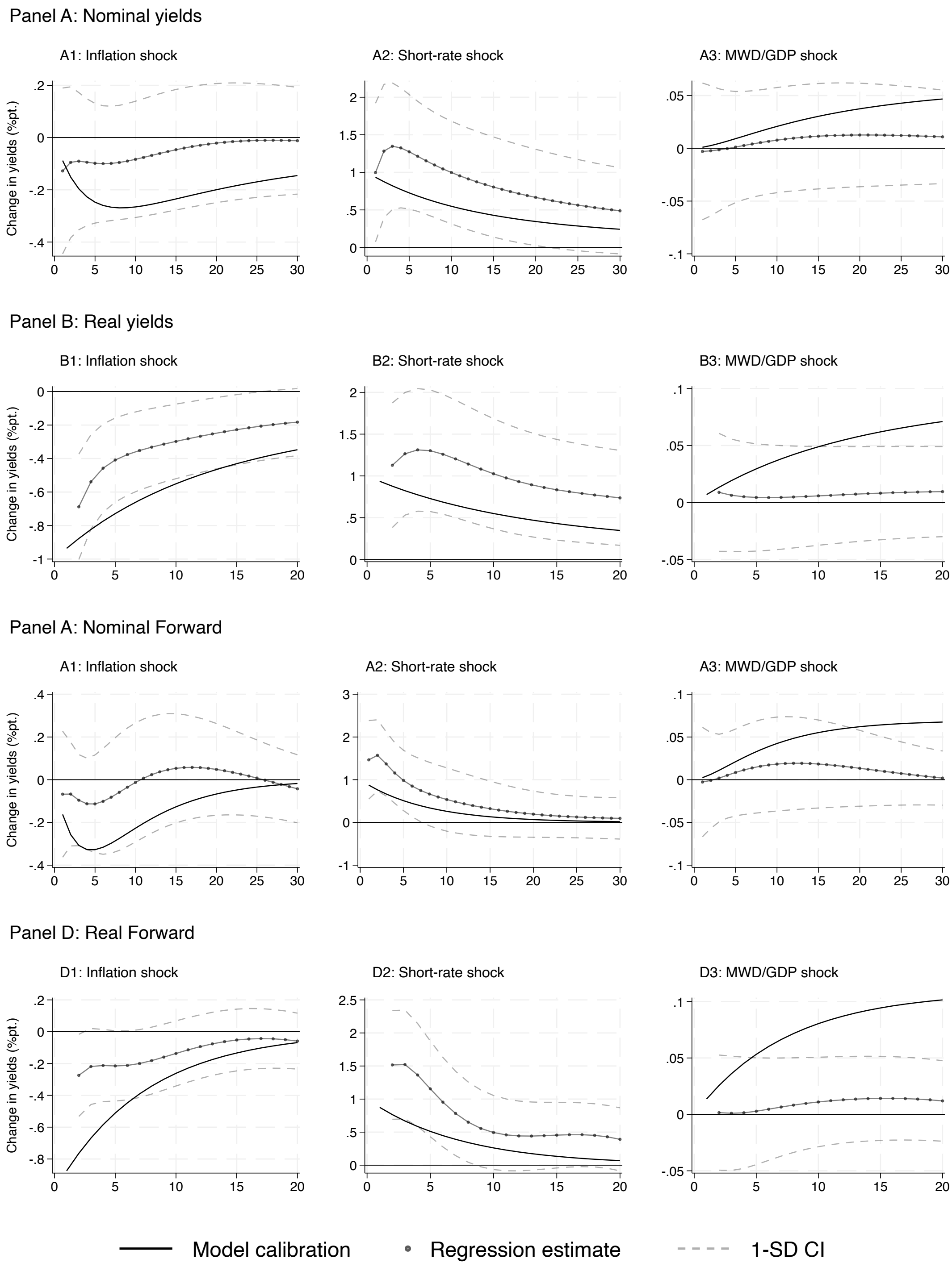

Figure D2:
Cross-section of the theoretical and empirical response of yields and forward rates – controlling for the VIX index.

Note: Plot of estimated $\boldsymbol{\beta}_f^{(\tau)}$ for $\tau \in (0, T_j]$ from (41) and (42), together with their theoretical counterpart in (26), (29), (32), (33).

D.3 4-variable recursive VAR

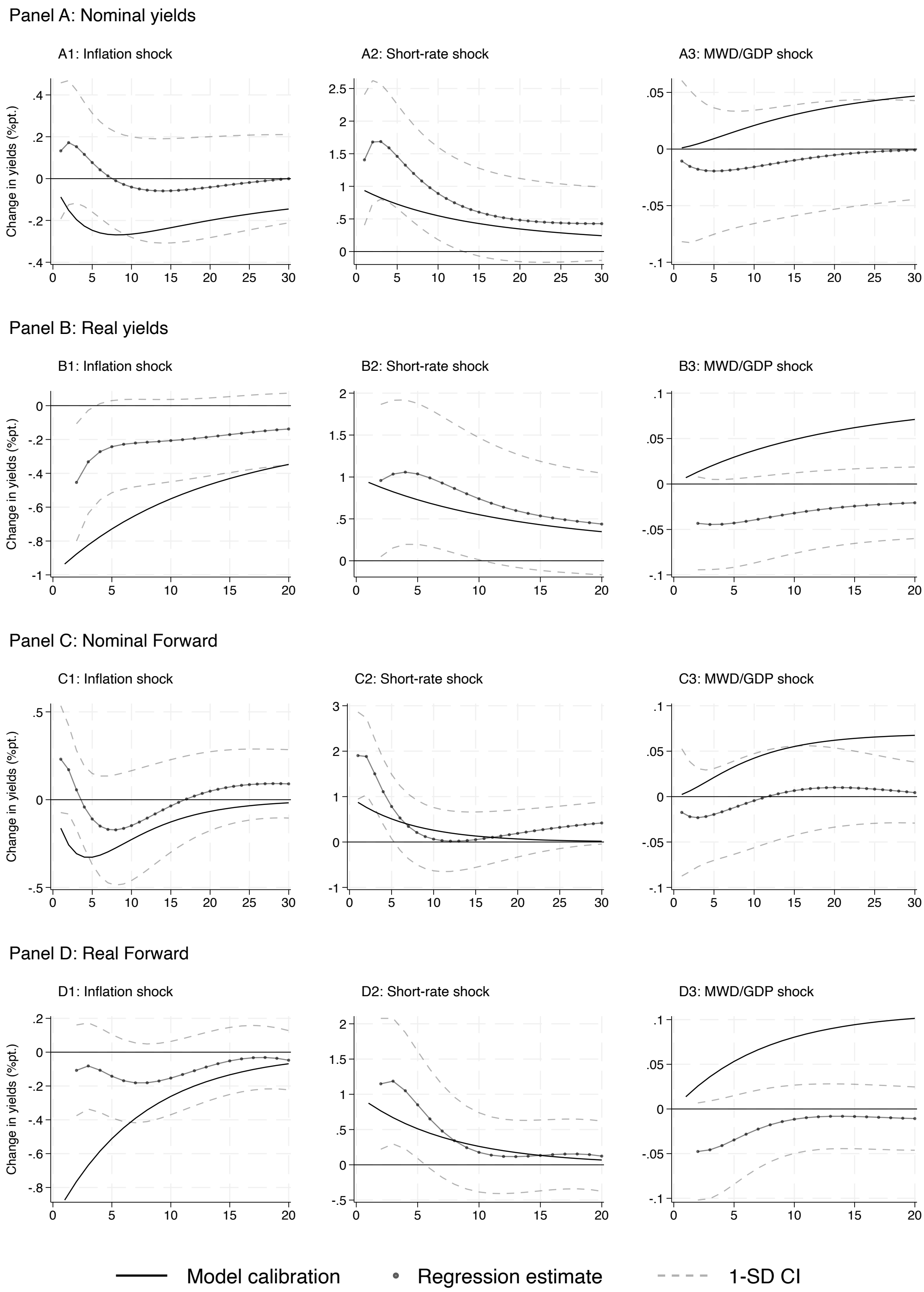

Figure D3:
Cross-section of the theoretical and empirical response of yields and forward rates – using shocks identified from a 4-variable VAR

Note: Plot of estimated $\boldsymbol{\beta}_f^{(\tau)}$ for $\tau \in (0, T_j]$ from (39) and (40) using public supply available (net of Federal holdings), together with their theoretical counterpart in (26), (29), (32), (33)

References

Backus, D., Foresi, S., and Telmer, C. (2001). Affine term structure models and the forward premium anomaly. *The Journal of Finance*, 56:279 – 304.

Bekaert, G. and Hodrick, R. (2001). Expectations hypotheses tests. *Journal of Finance*, 56(4):1357–1394.

Bernanke, B. S., Reinhart, V., and Sack, B. P. (2004). Monetary policy alternatives at the zero bound: an empirical assessment. Finance and Economics Discussion Series 2004-48, Board of Governors of the Federal Reserve System (U.S.).

Buraschi, A. and Jiltsov, A. (2005). Inflation risk premia and the expectations hypothesis. *Journal of Financial Economics*, 75(2):429–490.

Campbell, J. Y. and Shiller, R. J. (1991). Yield spreads and interest rate movements: A bird's eye view. *The Review of Economic Studies*, 58(3):495–514.

Campbell, J. Y., Shiller, R. J., and Viciera, L. M. (2009). Understanding Inflation-Indexed Bond Markets. *Brookings Papers on Economic Activity*, 2009:79–120.

Centre for Research in Security Prices (2024). CRSP. `https://www.crsp.org`. Accessed: 2024-04-28.

Christensen, J., Lopez, J., and Rudebusch, G. (2010). Inflation expectations and risk premiums in an arbitrage-free model of nominal and real bond yields. *Journal of Money, Credit and Banking*, 42(s1):143–178.

Christiano, L. J., Eichenbaum, M., and Evans, C. L. (1999). Monetary policy shocks: What have we learned and to what end? In Taylor, J. B. and Woodford, M., editors, *Handbook of Macroeconomics*, volume 1 of *Handbook of Macroeconomics*, chapter 2, pages 65–148. Elsevier.

Cochrane, J. H. and Piazzesi, M. (2005). Bond risk premia. *American Economic Review*, 95(1):138–160.

Cox, J. C., E, Ingersoll, J., and Ross, S. (1985). A Theory of the Term Structure of Interest Rates. *Econometrica*, 53(2):385–407.

Culbertson, J. M. (1957). The Term Structure of Interest Rates. *The Quarterly Journal of Economics*, 71(4):485–517.

Cúrdia, V. and Woodford, M. (2011). The central-bank balance sheet as an instrument of monetarypolicy. *Journal of Monetary Economics*, 58(1):54–79. Carnegie-Rochester Conference Series on Public Policy: The Future of Central Banking April 16-17, 2010.

Dai, Q. and Singleton, K. J. (2000). Specification Analysis of Affine Term Structure Models. *The Journal of Finance*, 55(5):1943–1978. Publisher: [American Finance Association, Wiley].

D'Amico, S., Kim, D. H., and Wei, M. (2008). Tips from TIPS: the informational content of Treasury Inflation-Protected Security prices. BIS Working Papers 248, Bank for International Settlements.

Duffee, G. (2002). Term premia and interest rate forecasts in affine models. *Journal of Finance*, 57:405–443.

Duffie, D. and Kan, R. (1996). A yield-factor model of interest rates. *Mathematical Finance*, 6(4):379–406.

D'Amico, S. and King, T. B. (2013). Flow and stock effects of large-scale treasury purchases: Evidence on the importance of local supply. *Journal of Financial Economics*, 108(2):425–448.

Fama, E. and Bliss, R. R. (1987). The information in long-maturity forward rates. *American Economic Review*, 77(4):680–92.

Federal Reserve Bank of St. Louis (2024). FRED Database. `https://fred.stlouisfed.org/`. Accessed: 2024-04-28.

Fleckenstein, M., Longstaff, F. A., and Lustig, H. (2010). Why does the treasury issue tips? the tips-treasury bond puzzle. Working Paper 16358, National Bureau of Economic Research.

Garbade, K. D. and Rutherford, M. (2007). Buybacks in Treasury cash and debt management. Staff Reports 304, Federal Reserve Bank of New York.

Gertler, M. and Karadi, P. (2011). A model of unconventional monetary policy. *Journal of Monetary Economics*, 58(1):17–34.

Gorodnichenko, Y., Ray, W., and Droste, M. (2023). The Effects of Quantitative Easing: Taking a Cue from Treasury Auctions. NBER Working Papers 24122, National Bureau of Economic Research, Inc.

Greene, W. H. (2020). *Econometric analysis*. Pearson, Harlow, England, eight, global edition. edition.

Greenwood, R., Hanson, S., and Vayanos, D. (2023). Supply and Demand and the Term Structure of Interest Rates. NBER Working Papers 31879, National Bureau of Economic Research, Inc.

Greenwood, R. and Vayanos, D. (2014). Bond Supply and Excess Bond Returns. *The Review of Financial Studies*, 27(3):663–713. Publisher: [Oxford University Press, The Society for Financial Studies].

Gürkaynak, R., Sack, B. P., and Wright, J. (2006). The U.S. Treasury yield curve: 1961 to the present. Finance and Economics Discussion Series 2006-28, Board of Governors of the Federal Reserve System (U.S.).

Gürkaynak, R., Sack, B. P., and Wright, J. (2008). The TIPS yield curve and inflation compensation. Finance and Economics Discussion Series 2008-05, Board of Governors of the Federal Reserve System (U.S.).

Gürkaynak, R. S. and Wright, J. H. (2012). Macroeconomics and the term structure. *Journal of Economic Literature*, 50(2):331–67.

He, Z. and Krishnamurthy, A. (2013). Intermediary asset pricing. *American Economic Review*, 103(2):732–70.

Krishnamurthy, A. and Vissing-Jorgensen, A. (2012). The aggregate demand for treasury debt. *Journal of Political Economy*, 120(2):233–267.

Lütkepohl, H. (2005). *New Introduction to Multiple Time Series Analysis*. Springer.

Mishkin, F. S. and Wright, J. H. (2009). Understanding inflation-indexed bond markets. comments and discussion. *Brookings Papers on Economic Activity*, 2009:121–138.

Modigliani, F. and Sutch, R. (1966). Innovations in interest rate policy. *The American Economic Review*, 56(1/2):178–197.

Newey, W. K. and West, K. D. (1987). A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix. *Econometrica*, 55(3):703–708.

Piazzesi, M. (2003). Affine term structure models. *Handbook of Financial Econometrics, Vol 1*, 1.

Piazzesi, M. and Schneider, M. (2006). Equilibrium Yield Curves. NBER Working Papers 12609, National Bureau of Economic Research, Inc.

Ray, W. (2019). Monetary Policy and the Limits to Arbitrage: Insights from a New Keynesian Preferred Habitat Model. 2019 Meeting Papers 692, Society for Economic Dynamics.

Ray, W., Droste, M., and Gorodnichenko, Y. (2024). Unbundling Quantitative Easing: Taking a Cue from Treasury Auctions. *Journal of Political Economy*, pages 000–000. Publisher: The University of Chicago Press.

Sims, C. A. (1980). Macroeconomics and reality. *Econometrica*, 48(1):1–48.

Sims, E. and Wu, J. C. (2021). Evaluating central banks' tool kit: Past, present, and future. *Journal of Monetary Economics*, 118:135–160.

Stock, J. H. and Watson, M. W. (2001). Vector autoregressions. *Journal of Economic Perspectives*, 15(4):101–115.

Swanson, E. (2011). Let's twist again: A high-frequency event-study analysis of operation twist and its implications for qe2. *Brookings Papers on Economic Activity*, 42(1 (Spring)):151–207.

Taylor, J. B. (1993). Discretion versus policy rules in practice. *Carnegie-Rochester Conference Series on Public Policy*, 39:195–214.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188.

Vayanos, D. and Vila, J.-L. (2021). A Preferred-Habitat Model of the Term Structure of Interest Rates. *Econometrica*, 89(1):77–112.

Wachter, J. A. (2006). A consumption-based model of the term structure of interest rates. *Journal of Financial Economics*, 79(2):365–399.