9 Preferred Habitats in the Term Structure

In this section, we present the seminal work of Vayanos and Vila (2021), who formalise the *preferred habitat* theory of the yield curve first hypothesised in Modigliani and Sutch (1966). Their model consists of two agents: preferred habitat investors who propagate local demand in a subset of the yield curve, and arbitrageurs that choose a portfolio of bonds across the yield curve.

9.1 Fixed Supply

We begin with the simplest version of the model, with fixed local demand/supply for each maturity τ. The fixed supply case provides a solid basis for understanding the mechanics of the Vayanos and Vila model and produces many of the important results.

9.1.1 Affine spot rates

As with the Vasicek model that we looked at earlier, we assume the short rate, r_t follows a one factor Ornstein-Uhlenbeck process

$$\mathrm{d}r_t = -\kappa\,(r_t - \bar{r})\,\mathrm{d}t + \sigma\,\mathrm{d}B_t \tag{9.1}$$

with mean reversion parameter, κ and variance σ^2.

9.1.2 Arbitrageur preferences

Arbitrageurs choose positions, $\{x_t^{(\tau)}\}_{\tau\in(0,T]}$ to maximise mean-variance utility over instantaneous changes in wealth

$$U_t = \mathbb{E}_t[\mathrm{d}W_t] - \frac{a}{2}\,\mathrm{var}_t(\mathrm{d}W_t) \tag{9.2}$$

subject to their budget constraint

$$\mathrm{d}W_t = W_t r_t\,\mathrm{d}t + \int_0^T x_t^{(\tau)} \left(\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} - r_t\,\mathrm{d}t \right) \mathrm{d}\tau \tag{9.3}$$

9.1.3 Equilibrium term structure

We conjecture that equilibrium log prices are affine functions of r_t, in the form

$$-\log P_t^{(\tau)} = A(\tau) r_t + C(\tau) \tag{9.4}$$

that is, there exists two functions, $A : (0,T] \to \mathbb{R}$, $C : (0,T] \to \mathbb{R}$ that depend solely on maturity. By Itô's Lemma, it holds that

$$\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} = \mu_t^{(\tau)} \mathrm{d}t - A(\tau)\,\sigma\,\mathrm{d}B_t \tag{9.5}$$

where

$$\mu_t^{(\tau)} := A'(\tau)\,r_t + C'(\tau) + A(\tau)\,\kappa\,(r_t - \bar{r}) + \frac{1}{2}\sigma^2 A(\tau)^2 \tag{9.6}$$

Theorem 22 *The arbitrageur's first order condition is*

$$\mu_t^{(\tau)} - r_t = -A(\tau)\lambda_t \tag{9.7}$$

where λ_t is the price of risk, defined by

$$\lambda_t := -a\sigma^2 \int_0^T A(\tau)\, x_t^{(\tau)}\, \mathrm{d}\tau \tag{9.8}$$

Proof. Substitute (9.5) into the budget constraint (9.3) to yield

$$\mathrm{d}W_t = \left[W_t r_t + \int_0^T x_t^{(\tau)} \left(\mu_t^{(\tau)} - r_t \right) \mathrm{d}\tau \right] \mathrm{d}t - \sigma \left[\int_0^T x_t^{(\tau)} A(\tau)\, \mathrm{d}\tau \right] \mathrm{d}B_t \tag{9.9}$$

Substituting the expectation and variance of $\mathrm{d}W_t$ yields the unconstrained maximisation problem

$$\max_{\{x_t^{(\tau)}\}_{\tau \in (0,T]}} U_t = W_t r_t + \int_0^T x_t^{(\tau)} \left(\mu_t^{(\tau)} - r_t \right) \mathrm{d}\tau - \frac{a\sigma^2}{2} \left[\int_0^T x_t^{(\tau)} A(\tau)\, \mathrm{d}\tau \right]^2 \tag{9.10}$$

Taking first order conditions with respect to $x_t^{(\tau)}$ yields the result. □

9.1.4 Solution

Next, we impose market clearing, that is $x_t^{(\tau)} = s_t^{(\tau)}$. Substituting market clearing and the definition for $\mu_t^{(\tau)}$ yields

$$\begin{aligned} A'(\tau)\,r_t + C'(\tau) + A(\tau)\,\kappa\,(r_t - \bar{r}) + \frac{1}{2}\sigma^2 A(\tau)^2 - r_t \\ = a\sigma^2 A(\tau) \int_0^T A(\tau)\, s_t^{(\tau)}\, \mathrm{d}\tau \end{aligned} \tag{9.11}$$

Identifying linear terms in r_t yields the ODE

$$A'(\tau) + \kappa A(\tau) - 1 = 0 \tag{9.12}$$

which is solved with the terminal condition $A(0) = 0$. Similarly for $C(\tau)$, having solved for $A(\tau)$ identifying linear constant terms yields the ODE

$$C'(\tau) - A(\tau)\left(\kappa\bar{r} + a\sigma^2 \int_0^T A(\tau)s_t^{(\tau)}\,\mathrm{d}\tau\right) + \frac{1}{2}\sigma^2 A(\tau)^2 = 0$$

with terminal condition $C(0) = 0$. The terminal conditions are derived by noting our bond pays one at maturity; to achieve this, the exponential coefficient must therefore be equal to zero.

Theorem 23 *The solution for $A(\tau)$ is*

$$A(\tau) = \frac{1 - e^{-\kappa\tau}}{\kappa} \tag{9.13}$$

Proof. Using the integrating factor method, we multiply both sides by $e^{\kappa\tau}$ to get

$$\frac{\mathrm{d}}{\mathrm{d}\tau} e^{\kappa\tau} A(\tau) = e^{\kappa\tau}$$

Integrating from zero to τ yields the result. □

Theorem 24 *The solution for $C(\tau)$ is*

$$C(\tau) = \left(\kappa\bar{r} + a\sigma^2 \int_0^T A(\tau)s_t^{(\tau)}\,\mathrm{d}\tau\right)\int_0^\tau A(u)\,\mathrm{d}u - \frac{\sigma^2}{2}\int_0^\tau A(u)^2\,\mathrm{d}u \tag{9.14}$$

Proof. Integrating (9.13) from zero to τ, and rearranging for $C(\tau)$ yields the result. □

9.1.5 Shocks to short rate

The *percentage change* in bond prices with respect to a change in short rates is given by

$$\frac{\partial \log P_t^{(\tau)}}{\partial r_t} = -A(\tau) \tag{9.15}$$

Theorem 25 *The percentage change in bond prices with respect to a change in short rates is*

(1) negative for all $\tau \in (0, T]$

(2) decreasing in τ (magnitude increasing)

Proof. For (1), note that $A(\tau)$ is positive for all τ as $\kappa > 0$ is assumed positive, and $e^{-x} < 1$ for any $x > 0$. For (2), the derivative, $A'(\tau) = e^{-\kappa\tau} > 0$ for all $\tau \in \mathbb{R}$, thus the magnitude of the percent change is increasing in τ. □

The *change* in yields with respect to a change in short rates is given by

$$\frac{\partial y_t^{(\tau)}}{\partial r_t} = \frac{A(\tau)}{\tau} \tag{9.16}$$

Theorem 26 *The change in yields with respect to a change in short rates is*

(1) positive for all $\tau \in (0, T]$

(2) decreasing in τ *(magnitude decreasing)*

Proof. For (1), we showed in the proof for the previous theorem that $A(\tau) > 0$ for all $\tau > 0$; the result therefore follows trivially. For (2), note the cross partial is

$$\frac{\partial^2 y_t^{(\tau)}}{\partial \tau\, \partial r_t} = \frac{A'(\tau)\tau - A(\tau)}{\tau^2} = \frac{(1+\kappa\tau)e^{-\kappa\tau} - 1}{\tau^2} \tag{9.17}$$

Note $(1+\kappa\tau)\, e^{-\kappa\tau} \leq 1$ from the elementary inequality $1 + x \leq e^x$; thus the cross partial is negative and the statement is proved. □

9.1.6 Shocks to supply and global price pressure

Looking at our solution, we can see that supply affects yields through $C(\tau)$ in (); in particular, it acts through the duration weighted ‘supply portfolio’, defined as $\int_0^T A(\tau)\, s_t^{(\tau)}\, \mathrm{d}\tau$.

The response of yields to an unanticipated, permanent, positive shock to the supply of bonds with maturity τ' can be easily verified as

$$\frac{\partial y_t^{(\tau)}}{\partial s_t^{(\tau')}} = \frac{a\sigma^2 A(\tau')}{\tau} \int_0^\tau A(u)\, \mathrm{d}u \tag{9.18}$$

$$= a\sigma^2 A(\tau')\, \frac{e^{-\kappa\tau} + \tau\kappa - 1}{\tau\kappa^2} \tag{9.19}$$

$$= a\sigma^2 A(\tau') \left(\frac{1}{\kappa} - \frac{1 - e^{-\kappa\tau}}{\tau\kappa^2} \right) \tag{9.20}$$

Theorem 27 *The change in yields with respect to a change in supply in maturity* τ' *is*

(1) positive for all $\tau \in (0, T]$

(2) increasing in τ *(magnitude increasing)*

Proof. For (1), note that since $\kappa\tau > 0$, it holds that $0 < 1 - e^{-\kappa\tau} < 1$, thus

$$\frac{1}{\kappa} > \frac{1}{\kappa^2} > \frac{1}{\kappa^2\tau} > \frac{1 - e^{-\kappa\tau}}{\kappa^2\tau} > 0 \tag{9.21}$$

The brackets term in () is therefore positive, and as we showed the preceding constants are positive, the whole of () is positive.

For (2), the cross partial is

$$\frac{\partial^2 y_t^{(\tau)}}{\partial\tau\,\partial s_t^{(\tau')}} = -\frac{a\sigma^2 A(\tau')}{\kappa}\frac{\partial^2 y_t^{(\tau)}}{\partial\tau\,\partial r_t} \tag{9.22}$$

In Theorem 26 above, we showed that the cross partial of yields on the short rate and τ is negative; this implies the whole expression above is positive. □

The last theorem has an important implication, namely in the fixed supply equilibrium, supply shocks in one part of the yield curve have a positive effect on the entire term structure. This is due to the fact yields of all maturities are dependent on the duration weighted supply portfolio; supply shocks in one part of the yield curve distort the supply portfolio duration and thus yields across the term structure.

9.2 Elastic Supply

In this part, we extend the fixed supply model to a model with market impact to arbitrageur trading, i.e. supply is responsive to price.

9.2.1 Preferred habitat investors

We introduce these so called 'preferred habitat' investors. We assume these investors have a log-linear demand function

$$z_t^{(\tau)} = -\alpha(\tau)\log P_t^{(\tau)} - \beta(\tau) \tag{9.23}$$

where $\alpha, \beta : (0, T] \to \mathbb{R}$ are deterministic functions of maturity.

Without arbitrageurs, the prevailing equilibrium yield would be completely determined by local demand from z_t, given by

$$y_t^{(\tau)} = \frac{\beta(\tau)}{\alpha(\tau)\,\tau} \tag{9.24}$$

The yield is disconnected from the short rate.

9.2.2 Equilibrium term structure

The analysis concerning arbitrageurs carries over to this section; instead we modify the market clearing condition to

$$x_t^{(\tau)} + z_t^{(\tau)} = 0 \tag{9.25}$$

() states that the sum of demand from arbitrageurs and preferred habitat investors for bonds of maturity $\tau \in (0, T]$ must be zero. Substituting into λ_t, the price of risk we have

$$
\begin{aligned}
\lambda_t &= a\sigma^2 \int_0^T A(\tau)\, z_t^{(\tau)}\, \mathrm{d}\tau && (9.26)\\
&= a\sigma^2 \int_0^T A(\tau) \left[-\alpha(\tau) \log P_t^{(\tau)} - \beta(\tau)\right] \mathrm{d}\tau && (9.27)\\
&= a\sigma^2 \int_0^T A(\tau) \left[\alpha(\tau)(A(\tau) r_t + C(\tau)) - \beta(\tau)\right] \mathrm{d}\tau && (9.28)
\end{aligned}
$$

where the last line follows from the affine conjecture. Substituting into the arbitrageurs first order condition, we have

$$
\begin{aligned}
&A'(\tau)\, r_t + C'(\tau) + A(\tau)\, \kappa\, (r_t - \bar{r}) + \frac{1}{2}\sigma^2 A(\tau)^2 - r_t\\
&\qquad = -a\sigma^2 A(\tau) \int_0^T A(\tau) \left[\alpha(\tau)(A(\tau) r_t + C(\tau)) - \beta(\tau)\right] \mathrm{d}\tau \qquad (9.29)
\end{aligned}
$$

As with the fixed supply case, we identify linear terms in r_t which yields the ODE

$$
A'(\tau) + A(\tau) \left(\kappa + a\sigma^2 \int_0^T \alpha(\tau) A(\tau)^2\, \mathrm{d}\tau\right) - 1 = 0 \qquad (9.30)
$$

and linear in constants yields the ODE

$$
C'(\tau) - A(\tau) \left(\kappa \bar{r} + a\sigma^2 \int_0^T A(\tau) \left[\beta(\tau) - \alpha(\tau) C(\tau)\right] \mathrm{d}\tau\right) + \frac{1}{2}\sigma^2 A(\tau)^2 = 0 \qquad (9.31)
$$

Equations () and () are again solved with the initial conditions $A(0) = C(0) = 0$. However, in contrast to the fixed supply case, a complicating feature of elastic supply is that the coefficient of $A(\tau)$ depends on an integral involving the functions $A(\tau), C(\tau)$. We proceed by:

1. taking the integrals as given and solving () and () as linear first order ODEs with constant coefficients, like we did before.
2. checking the solution is consistent with the value of the integrals.

Theorem 28 *The solution for $A(\tau)$ is*

$$
A(\tau) = \frac{1 - e^{-\kappa^* \tau}}{\kappa^*} \qquad (9.32)
$$

where κ^* is the risk-neutral counterpart of κ, which uniquely satisfies the fixed point condition

$$
\kappa^* = \kappa + a\sigma^2 \int_0^T \alpha(\tau) \left(\frac{1 - e^{-\kappa^* \tau}}{\kappa^*}\right)^2 \mathrm{d}\tau \qquad (9.33)
$$

Proof. Using the integrating factor method, we multiply both sides by $e^{\kappa^*\tau}$, with the initial definition

$$\kappa^* = \kappa + a\sigma^2 \int_0^T \alpha(\tau) A(\tau)^2 \, \mathrm{d}\tau \tag{9.34}$$

to get

$$\frac{\mathrm{d}}{\mathrm{d}\tau} e^{\kappa^*\tau} A(\tau) = e^{\kappa^*\tau} \tag{9.35}$$

Integrating from zero to τ yields (9.32).

To show κ^* has a *unique* solution, note the left side of (9.33) is increasing from zero to ∞. On the other hand, the right side of (9.33) is

- decreasing in κ^* - the derivative is

$$\begin{aligned} &\frac{\partial}{\partial \kappa^*} \int_0^T \alpha(\tau) \left(\frac{1 - e^{-\kappa^*\tau}}{\kappa^*} \right)^2 \mathrm{d}\tau \\ &\qquad = 2 \int_0^T \alpha(\tau) \, \frac{1 - e^{-\kappa^*\tau}}{\kappa^*} \, \frac{(1 + \kappa^*\tau) e^{-\kappa^*\tau} - 1}{\kappa^{*2}} \, \mathrm{d}\tau \end{aligned} \tag{9.36}$$

  As $(1 + \kappa^*\tau)\, e^{-\kappa^*\tau} \leq 1$ due to the elementary inequality $1 + x \leq e^x$, and the rest of the terms are positive, this implies the derivative is negative.
- is always greater than κ - this is because $\alpha(\tau)$ and $A(\tau)^2$ are always positive, and so the integral is positive.

Therefore κ^* reaches a unique equilibrium greater than κ. □

Theorem 29 *The solution for $C(\tau)$ is*

$$C(\tau) = \kappa^* \bar{r}^* \int_0^\tau A(u) \, \mathrm{d}u + \frac{\sigma^2}{2} \int_0^\tau A(u)^2 \, \mathrm{d}u \tag{9.37}$$

where $\bar{r}^*$ is the risk-neutral counterpart of $\bar{r}$ defined by

$$\kappa \bar{r}^* = \kappa \bar{r} + a\sigma^2 \int_0^T A(\tau) \left[\beta(\tau) - \alpha(\tau) C(\tau) \right] \mathrm{d}\tau \tag{9.38}$$

Proof. Integrate (9.31) from zero to τ. □

Although we omit the closed form expression for $\bar{r}^*$ here, we refer interested readers to Vayanos and Vila (2021), who derive an explicit expression for $\bar{r}^*$ dependent on κ^*.

9.2.3 Forward rates and the Expectations Hypothesis

Definition 15 (Forward rate) *The forward rate between time $\tau - \delta$ and τ, at time t is the average difference in log prices*

$$f_t^{(\tau-\delta,\tau)} = -\frac{1}{\delta}\left(\log P_t^{(\tau)} - \log P_t^{(\tau-\delta)}\right) \tag{9.39}$$

If we take the limit of the time interval δ, we obtain the forward rate for borrowing over an infinitesimally long interval.

Definition 16 (Instantaneous forward rate) *The instantaneous forward rate for time τ at time t is*

$$f_t^{(\tau)} = \lim_{\delta\to 0} f_t^{(\tau-\delta,\tau)} \tag{9.40}$$

Lemma 2 *Given our affine conjecture in (), the instantaneous forward rate can be expressed as*

$$f_t^{(\tau)} = -\frac{\partial \log P_t^{(\tau)}}{\partial \tau} = A'(\tau)\, r_t + C'(\tau) \tag{9.41}$$

Proof. The right hand side of (9.40) is just the definition of the derivative with $g(\tau) = \log P_t^{(\tau)}$, i.e.

$$g'(\tau) := \lim_{\delta\to 0} \frac{g(\tau) - g(\tau-\delta)}{\tau - (\tau-\delta)} \tag{9.42}$$

□

If the expectations hypothesis holds, then the instantaneous forward rate is the equal to the expected future short rate. However, as demand is price-elastic the forward under-reacts to changes in the short rate today. Arbitrageurs only partially transmit short rate shocks to the yield curve today, and thus the extent to which no-arbitrage holds depends on the willingness of arbitrageurs to take on carry trade risk through their risk-aversion a.

Theorem 30 *The forward rate under-reacts to shocks to the short rate relative to expectations; the extent depends on $\kappa^* - \kappa$.*

Proof. We first study the effect of a short rate shock on short rate expectations, $\mathbb{E}_t[r_{t+\tau}]$. In the Appendix, we show the solution to the Ornstein-Uhlenbeck process is given by

$$r_{t+\tau} = e^{-\kappa\tau} r_t + (1 - e^{-\kappa\tau})\, \bar{r} + \sigma \int_t^{t+\tau} e^{-\kappa(t+\tau-s)}\, \mathrm{d}W_s \tag{9.43}$$

and has conditional expectation

$$\mathbb{E}_t[r_{t+\tau}] = e^{-\kappa\tau} r_t + (1 - e^{-\kappa\tau})\, \bar{r} \tag{9.44}$$

The response to the short rate today is therefore

$$\frac{\partial\,\mathbb{E}_t[r_{t+\tau}]}{\partial r_t} = e^{-\kappa\tau} \tag{9.45}$$

On the other hand, using Lemma 2 the response of the forward rate is

$$\frac{\partial f_t^{(\tau)}}{\partial r_t} = A'(\tau) = e^{-\kappa^*\tau} \tag{9.46}$$

The ratio of the responses is given by

$$\frac{\frac{\partial f_t^{(\tau)}}{\partial r_t}}{\frac{\partial\,\mathbb{E}_t[r_{t+\tau}]}{\partial r_t}} = \frac{e^{-\kappa\tau}}{e^{-\kappa^*\tau}} = e^{(\kappa^*-\kappa)\tau} \tag{9.47}$$

The larger the ratio, the larger the extent of the under-reaction. □

9.2.4 Global supply effects

9.3 Stochastic Supply

In this section we follow Greenwood and Vayanos (2014) in examining the case when supply is stochastic, but inelastic. We assume supply, $s_t^{(\tau)}$ is affine in a single stochastic supply factor, β_t

$$s_t^{(\tau)} = \zeta(\tau) + \theta(\tau)\beta_t \tag{9.48}$$

where β follows a zero-mean Ornstein-Uhlenbeck process,

$$\mathrm{d}\beta_t = -\kappa_\beta\,\beta_t + \sigma_\beta\,\mathrm{d}B_{\beta,t} \tag{9.49}$$

For simplicity, we assume r_t and β_t are *uncorrelated*, but interested readers may refer to the appendix of Greenwood and Vayanos (2014) for the derivation of the correlated case.

9.3.1 Equilibrium term structure

With supply risk, we instead conjecture bond yields are affine in r_t and β_t, that is, there exist functions $A_r, A_\beta, C : (0,T] \to \mathbb{R}$ such that

$$-\log P_t^{(\tau)} = A_r(\tau)\,r_t + A_\beta(\tau)\beta_t + C(\tau) \tag{9.50}$$

By Itô's lemma, it holds that

$$\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} = \mu_t^{(\tau)} - A_r(\tau)\,\sigma_r\,\mathrm{d}B_{r,t} - A_\beta(\tau)\,\sigma_\beta\,\mathrm{d}B_{\beta,t} \tag{9.51}$$

where

$$\mu_t^{(\tau)} := A_r'(\tau)\, r_t + A'(\tau)\beta_t + C'(\tau) + A_r(\tau)\, \kappa_r (r_t - \bar{r}) + A_\beta(\tau)\, \kappa_\beta \beta_t \\ + \frac{1}{2}\sigma_r^2 A_r(\tau)^2 + \frac{1}{2}\sigma_\beta^2 A_\beta(\tau)^2 \tag{9.52}$$

A note on notation; as our model now has two stochastic factors, we denote the parameters associated with the short rate by a subscript r, where there was none previously.

Theorem 31 *The arbitrageur's first order condition is*

$$\mu_t^{(\tau)} - r_t = -A_r(\tau)\, \lambda_{r,t} - A_\beta(\tau)\, \lambda_{\beta,t} \tag{9.53}$$

where $\lambda_{j,t}$ is the price of risk for factor j defined as

$$\lambda_{j,t} := -a\sigma_j^2 \int_0^T A_j(\tau)\, x_t^{(\tau)}\, \mathrm{d}\tau \tag{9.54}$$

Proof. Substitute () into the budget constraint (9.3) to yield

$$\mathrm{d}W_t = \left[W_t r_t + \int_0^T x_t^{(\tau)} \left(\mu_t^{(\tau)} - r_t\right) \mathrm{d}\tau \right] \mathrm{d}t \\ - \left[\int_0^T A_r(\tau)\, x_t^{(\tau)}\, \mathrm{d}\tau \right] \sigma_r\, \mathrm{d}B_{r,t} - \left[\int_0^T A_\beta(\tau)\, x_t^{(\tau)}\, \mathrm{d}\tau \right] \sigma_\beta\, \mathrm{d}B_{\beta,t} \tag{9.55}$$

Substituting the mean and variance of $\mathrm{d}W_t$ yields an unconstrained maximisation problem

$$\max_{\{x_t^{(\tau)}\}_{\tau \in (0,T]}} \quad U_t = W_t r_t + \int_0^T x_t^{(\tau)} (\mu_t^{(\tau)} - r_t)\, \mathrm{d}\tau \\ - \frac{a\sigma_r^2}{2} \left[\int_0^T A_r(\tau)\, x_t^{(\tau)}\, \mathrm{d}\tau \right]^2 - \frac{a\sigma_\beta^2}{2} \left[\int_0^T A_\beta(\tau)\, x_t^{(\tau)}\, \mathrm{d}\tau \right]^2 \tag{9.56}$$

Taking first order conditions with respect to $x_t^{(\tau)}$ yields the result. □

9.3.2 Solution

Next we impose market clearing, that is $x_t^{(\tau)} = s_t^{(\tau)}$. Substituting market clearing into the risk prices leaves

$$\lambda_{j,t} = -a\sigma_j^2 \int_0^T A_j(\tau)\; [\zeta(\tau) + \theta(\tau)\beta_t]\, \mathrm{d}\tau \tag{9.57}$$

Although we omit writing the whole expression for the first order condition () as in the fixed and elastic case, one can easily verify that identifying linear terms in the variables below yields the following ODEs.

- *short rate*, r_t

$$A_r'(\tau) + \kappa_r A(\tau) - 1 = 0 \tag{9.58}$$

- *supply factor*, β_t

$$A_\beta'(\tau) + A_\beta(\tau)\left(\kappa_\beta - a\sigma_\beta^2 \int_0^T A_\beta(\tau)\,\theta(\tau)\,\mathrm{d}\tau\right) = a\sigma_r^2 A_r(\tau) \int_0^T A_r(\tau)\,\theta(\tau)\,\mathrm{d}\tau \tag{9.59}$$

- *constants*,

$$\begin{aligned} &C'(\tau) - \kappa_r \bar{r} A_r(\tau) + \frac{1}{2}\sigma_r^2 A_r(\tau)^2 + \frac{1}{2}\sigma_\beta^2 \\ &\qquad = a\sigma_r^2 A_r(\tau) \int_0^T A_r(\tau)\,\zeta(\tau)\,\mathrm{d}\tau + a\sigma_\beta^2 A_\beta(\tau) \int_0^T A_\beta(\tau)\,\zeta(\tau)\,\mathrm{d}\tau \end{aligned} \tag{9.60}$$

which are all solved with terminal condition $A_r(0) = A_\beta(0) = C(0) = 0$. Similar to the elastic case, note () is again an ODE where the coefficient of $A_\beta(\tau)$ depends on an integral involving the functions $A_r(\tau), A_\beta(\tau)$. In addition, we have a linear system of ODEs; fortunately here it can be solved recursively.

Theorem 32 *The solution for $A_r(\tau), A_\beta(\tau)$ is given by*

$$A_r(\tau) = \frac{1 - e^{-\kappa_r \tau}}{\kappa_r} \tag{9.61}$$

$$A_\beta(\tau) = \frac{L_r}{\kappa_r}\left(\frac{1 - e^{-\kappa_\beta^* \tau}}{\kappa_\beta^*} - \frac{e^{-\kappa_r \tau} - e^{-\kappa_\beta^* \tau}}{\kappa_\beta^* - \kappa_r}\right) \tag{9.62}$$

where L_r is a scalar constant defined by

$$L_r := a\sigma_r^2 \int_0^T \frac{1 - e^{-\kappa \tau}}{\kappa}\,\theta(\tau)\,\mathrm{d}\tau \tag{9.63}$$

and κ_β^* solves the fixed point condition

$$\kappa_\beta^* = \kappa_\beta - \frac{a\sigma_\beta^2 Z}{\kappa_r} \int_0^T \left(\frac{1 - e^{-\kappa_\beta^* \tau}}{\kappa_\beta^*} - \frac{e^{-\kappa \tau} - e^{-\kappa_\beta^* \tau}}{\kappa_\beta^* - \kappa}\right)\theta(\tau)\,\mathrm{d}\tau \tag{9.64}$$

Proof. The ODE for $A_r(\tau)$ in () is the same as the fixed supply case, and so we obtain the same solution.

For $A_\beta(\tau)$, we follow the same approach as in the elastic supply case. Initially define κ_β^* as

$$\kappa_\beta^* = \kappa_\beta - a\sigma_\beta^2 \int_0^T A_\beta(\tau)\,\theta(\tau)\,\mathrm{d}\tau \tag{9.65}$$

Given the solution to $A_r(\tau)$ and κ_β^* above, () becomes

$$A_\beta'(\tau) + \kappa_\beta^* A_\beta(\tau) = \frac{L_r}{\kappa_r}(1 - e^{-\kappa_r \tau}) \tag{9.66}$$

Multiplying by the integrating factor $e^{\kappa_\beta^* \tau}$, and integrating both sides from zero to τ yields

$$\begin{aligned} e^{\kappa_\beta^* \tau} A_\beta(\tau) &= \frac{L_r}{\kappa_r} \int_0^\tau e^{\kappa_\beta^* u} - e^{(\kappa_\beta^* - \kappa_r)u} \, \mathrm{d}u && (9.67) \\ &= \frac{L_r}{\kappa_r} \left(\frac{e^{\kappa_\beta^* \tau} - 1}{\kappa_\beta^*} - \frac{e^{(\kappa_\beta^* - \kappa_r)\tau} - 1}{\kappa_\beta^* - \kappa_r} \right) && (9.68) \end{aligned}$$

which implies

$$A_\beta(\tau) = \frac{L_r}{\kappa_r} \left(\frac{1 - e^{-\kappa_\beta^* \tau}}{\kappa_\beta^*} - \frac{e^{-\kappa_r \tau} - e^{-\kappa_\beta^* \tau}}{\kappa_\beta^* - \kappa_r} \right) \tag{9.69}$$

□

9.4 Extensions

9.4.1 Real and nominal rates

Zhao (2024) models the real and nominal term structure jointly, with arbitrageurs selecting a portfolio over nominal and real bonds with mean-variance preferences over real wealth.

9.5 Evidence