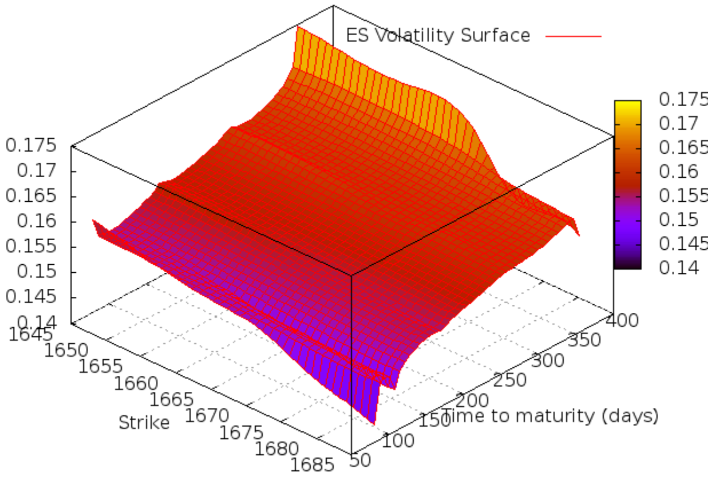
Mathematics of Finance and Valuation



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Introduction

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Chapter 1

Stochastic Processes

1.1 Filtrations and Stopping Times

Definition 1.1.1: Stochastic Process

A *stochastic process* is a parameterised collection of random variables $\{X_t\}_{t\in\mathcal{T}}$ defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathcal{T}$ is the time set, and can either be:

- **discrete**, if $\mathcal{T} = \{0, 1, 2, \dots\}$
- **continuous**, if $\mathcal{T} = [0, T]$ or $\mathcal{T} = [0, \infty)$

There are two ways of viewing a stochastic process:

- as a random variable; for $\omega \in \Omega,\ \omega \to X_t(\omega)$
- as a sample path/realisation; for $t \in \mathcal{T},\ t \to X(t)$

1.1.1 Filtrations

The next concept is filtrations, which model flows of information:

Definition 1.1.2: Filtration

A *filtration* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of σ-algebras $(\mathcal{F}_t)_{t\in\mathcal{T}}$ s.t. for all $t \in \mathcal{T}$:

(a) $\mathcal{F}_t \subseteq \mathcal{F}$

(b) $\mathcal{F}_s \subseteq \mathcal{F}_t$ when $s \leq t$; i.e. no information is lost over time.

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)$ is denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, and called a *filtered probability space*.

In the rest of this course we assume that filtrations satisfy so called *usual conditions*.

Property 1.1.3: Usual conditions

We consider two *usual conditions*:

(a) **right continuity**: for all $t \in \mathcal{T}$,

$$\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$$

(b) **completeness**: for all $t \in \mathcal{T}$,

$$\mathcal{N} \subseteq \mathcal{F}_0 \subset \mathcal{F}_t$$

where $\mathcal{N}$ is the $\mathbb{P}$-null set defined as:

$$\mathcal{N} := \{A \subset \Omega : \exists B \in \mathcal{F} \text{ s.t. } A \subseteq B \text{ and } \mathbb{P}(B) = 0\}$$

Definition 1.1.4: Natural filtration

For a general stochastic process X, the *natural filtration*, $(\mathcal{F}_t^X)$ is defined as

$$\mathcal{F}_t^X := \sigma\left(\{X_s : s \leq t, t \in \mathcal{T}\}\right)$$

Note natural filtrations often fail the usual conditions, so we need to augment them in practice.

Definition 1.1.5: Adapted process

A stochastic process X_t is *adapted* to the filtration $(\mathcal{F}_t)$, if for any $t \in \mathcal{T}$, the random variable X_t is $\mathcal{F}_t$ measurable.

Property 1.1.6

If X_t is $(\mathcal{F}_t)$-adapted, f is continuous, then $f(X_t)$ is $(\mathcal{F}_t)$-adapted.

Proof

Continuous (deterministic) functions preserve measurability. ■

1.1.2 Stopping times

Definition 1.1.7: Random time

A *random time*, τ is any random variable with values in $\mathcal{T} \cup \{\infty\}$. We use random times to denote the time at which a given random event occurs. $\{\tau = \infty\}$ represents the event that the random event never occurs.

Definition 1.1.8: Stopping time

Given a filtration $(\mathcal{F}_t)$, we say that a map $\tau : \Omega \to \mathcal{T}$ is an $(\mathcal{F}_t)$-*stopping time* if for all $t \in \mathcal{T}$

$$\tau^{-1}([0,t]) = \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$$

that is, the random time for an event occurring, $\tau \in [0,t]$ before time t must solely depend on the information known before t, but not after.

For a given stopping time $\leq t$, the information in $(\mathcal{F}_t)$ is enough to know whether the event has occurred or not.

Note that the filtration is essential for the defintion of stopping times. A random time can be a stopping time w.r.t. to $(\mathcal{F}_t)$ but not some other filtration $(\mathcal{G}_t)$.

Example.

> Suppose that τ_1 and τ_2 are two $(\mathcal{F}_t)$ stopping times. Then the random time τ defined by
>
> $$\tau := \min\{\tau_1, \tau_2\}$$
>
> is a $(\mathcal{F}_t)$ stopping time.

Proof. We have:

$$\{\tau > t\} := \{\tau > t\} \cap \{\tau_2 > t\}$$

therefore:

$$\begin{aligned} \{\tau \leq t\} &= \{\tau > t\}^c \\ &= \{\tau_1 \leq t\} \cup \{\tau_2 \leq t\} \in \mathcal{F}_t \end{aligned}$$

where the last inclusion follows as τ_1 and τ_2 are both stopping times and in $\mathcal{F}_t$, and $\mathcal{F}_t$ is closed under countable unions. ☐

Example.

> Let X_t be a continuous-time stochastic process which is $(\mathcal{F}_t)$-adapted, and assume that
>
> $$\forall \omega \in \Omega,\ t \to X_t(\omega) \text{ is right-continuous}$$
>
> i.e. every sample path of X is right-continuous. Let $a \in \mathbb{R}$ and set
>
> $$\tau := \inf\{t \geq 0 : X_t > a\}$$
>
> We claim τ is a stopping time.

Proof. Since X_t is $(\mathcal{F}_t)$-adapted, then for all $t \in \mathbb{R}_+$

$$X_t^{-1}([a, \infty)) := \{\omega \in \Omega : X_t(\omega) > a\} \in \mathcal{F}_t$$

Also, by assumption the sample-paths of X_t is right continuous, therefore for every $\omega \in \Omega$

$$\sup_{s \in [0,t)} X_s(\omega) = \sup_{s \in [0,t) \cap \mathbb{Q}} X_s(\omega)$$

by Lemma 6.0.1. Now consider:

$$\begin{aligned} \{\omega \in \Omega : \tau(\omega) < t\} &= \{\omega \in \Omega : \sup_{s \in [0,t)} X_s(\omega) > a\} \\ &= \{\omega \in \Omega : \sup_{s \in [0,t) \cap \mathbb{Q}} X_s(\omega) > a\} \end{aligned}$$

$$= \underbrace{\bigcup_{s \in [0,t) \cap \mathbb{Q}}}_{\text{countable intersec.}} \{\omega \in \Omega : X_s(\omega) > a\} \in \mathcal{F}_t$$

since $\{\omega \in \Omega : X_s(\omega) > a\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$, for all $s \in [0,t)$. Furthermore,

$$\{\omega \in \Omega : \tau(\omega) \leq t\} = \bigcap_{s>t} \{\omega \in \Omega : \tau(\omega) < s\} \overbrace{\in}^{\text{by above}} \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$$

where the final equality is by usual condition (a). Therefore τ is a stopping time. ☐

Definition 1.1.9: Stopping time filtration

Let τ be an $(\mathcal{F}_t)$-stopping time. Then we define the *stopping time* σ-algebra $\mathcal{F}_\tau$ by:

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$$

Note $\mathcal{F}_\tau$ can be interpreted as all information we have at time τ.

Example.

> Suppose the filtration $(\mathcal{F}_t)$ satisfies the usual conditions. Let τ_1 and τ_2 be two $(\mathcal{F}_t)$-stopping times satisfying $\tau_1 \leq \tau_2$. Then:
>
> $$\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$$

Proof. Let $A \in \mathcal{F}_{\tau_1}$. We want to show this implies $A \in \mathcal{F}_{\tau_2}$. By definition of A, then

$$A \cap \{\tau_1 \leq t\} \in \mathcal{F}_t$$

As τ_2 is a stopping time, by definition

$$\{\tau_2 \leq t\} \in \mathcal{F}_t$$

As $\mathcal{F}_t$ is a σ-algebra, the intersection of the two above is also in $\mathcal{F}_t$

$$A \cap \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\} \in \mathcal{F}_t$$

But since $\tau_1 \leq \tau_2$, $\tau_2 \leq t \implies \tau_1 \leq t$ but not the reverse, thus $\{\tau_2 \leq t\} \subseteq \{\tau_1 \leq t\}$. The above equation is just:

$$A \cap \{\tau_2 \leq t\} \in \mathcal{F}_t$$

which means $A \in \mathcal{F}_{\tau_2}$ by definition of a stopping time filtration. ☐

Example.

> Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of $(\mathcal{F}_t)$-stopping times decreasing to a random time τ, i.e. $\tau_n \downarrow \tau$. Then
>
> (a) τ is an $(\mathcal{F}_t)$-stopping time
>
> (b) $\mathcal{F}_\tau = \bigcap_{n=1}^{\infty} \mathcal{F}_{\tau_n}$

Proof.

(a) Note that $\{\tau_n \leq t\}$ is an increasing sequence of sets by the previous example as τ_n is decreasing, more and more ω's satisfy $\tau_n(\omega) \leq t$. Since τ_n is a stopping time, it holds that for any $n \in \mathbb{N}$, $\{\tau_n \leq t\} \in \mathcal{F}_t$.

Therefore the countable intersection is is also in $\mathcal{F}_t$, that is:

$$\{\tau \leq t\} = \bigcup_{n=1}^{\infty} \{\tau_n \leq t\} \in \mathcal{F}_t$$

thus τ is an $(\mathcal{F}_t)$-stopping time.

(b) We first show $\bigcap_{n=1}^{\infty} \mathcal{F}_{\tau_n} \subseteq \mathcal{F}_t$. Let $A \in \bigcap_{n=1}^{\infty} \mathcal{F}_{\tau_n}$, we want to show this implies $A \in \mathcal{F}_\tau$. Note the assumption by definition of a stopping time σ-algebra means for all n

$$A \cap \{\tau_n \leq t\} \in \mathcal{F}_t$$

Therefore we can say that:

$$\begin{aligned} A \cap \{\tau \leq t\} &= A \cap \left(\bigcup_{n=1}^{\infty} \{\tau_n \leq t\} \right) \\ &= \bigcup_{n=1}^{\infty} A \cap \{\tau_n \leq t\} \in \mathcal{F}_t \end{aligned}$$

which implies $A \in \mathcal{F}_\tau$.

Next we show the converse, that $\mathcal{F}_\tau \subseteq \bigcap_{n=1}^{\infty} \mathcal{F}_{\tau_n}$. By the previous example, we have that

$$\tau \leq \tau_{n+1} \leq \tau_n \implies \mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_{n+1}} \subseteq \mathcal{F}_{\tau_n}$$

Therefore, $\mathcal{F}_\tau \subseteq \bigcap_{n=1}^{\infty} \mathcal{F}_{\tau_n}$ and combined with the last paragraph, this means:

$$\mathcal{F}_\tau = \bigcap_{n=1}^{\infty} \mathcal{F}_{\tau_n}$$

□

1.2 Martingales

Definition 1.2.1: Martingale

A process X on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *martingale* w.r.t. $((\mathcal{F}_t), \mathbb{P})$ if

(a) X is **adapted**

(b) $\mathbb{E}[|X_t|] < \infty$ (**integrable**)

(c) for all $s > t$, $\mathbb{E}[X_s | \mathcal{F}_t] = X_t$ (**martingale property**)

Thus, the best predictor for the future value of a martingale is its current value. In discrete time, we can substitute (c) with: for all $t = 0, 1, 2, \ldots$

$$\mathbb{E}[X_{t+1} | \mathcal{F}_t] = X_t$$

Proof. This follows from applying the Law of Iterated Expectation (and induction):

$$\mathbb{E}[X_{t+1} | \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[X_{t+k} | \mathcal{F}_{t+k-1}] | \mathcal{F}_t] = \mathbb{E}[X_{t+k-1} | \mathcal{F}_t] = \cdots = X_t$$

□

Lemma 1.2.2: Martingale construction

Consider $(\Omega, \mathcal{F}, \mathbb{P})$, with the filtration $(\mathcal{F}_t)$, and a random variable X with $\mathbb{E}[|X|] < \infty$. Define the stochastic process M_t as the best prediction of X:

$$M_t := \mathbb{E}[X_t|\mathcal{F}_t]$$

Then M_t is a martingale.

Proof for Lemma

We check the conditions:

(a) M_t is adapted from the definition of conditional expectation.

(b) By conditional Jensen's inequality,

$$\mathbb{E}[|M_t|] = \mathbb{E}[|\mathbb{E}[X|\mathcal{F}_t]|] \leq \mathbb{E}[\mathbb{E}[|X|\,|\mathcal{F}_t]] = \mathbb{E}[|X_t|] < \infty$$

(c) For all $s \geq t$,

$$\mathbb{E}[M_s|\mathcal{F}_t] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_s]\,|\mathcal{F}_t] = \mathbb{E}[X|\mathcal{F}_t] = M_t$$

■

Definition 1.2.3: Sub and Supermartingales

Instead of (c), if for all $s > t$,

- $\mathbb{E}[X_s|\mathcal{F}_t] \geq X_t$, then X is a *submartingale*
- $\mathbb{E}[X_s|\mathcal{F}_t] \leq X_t$, then X is a *supermartingale*

Lemma 1.2.4

If a supermartingale on $[0, T]$ satisfies $\mathbb{E}[X_T] = X_0$, then it is a martingale on $[0, T]$.

Proof for Lemma

Let X_t be a supermartingale. Let $t \in [0, T]$ s.t. $T > t > 0$. By the supermartingale property, $X_0 \geq \mathbb{E}[X_t|\mathcal{F}_0]$, thus taking expectations yields:

$$\mathbb{E}[X_0] \geq \mathbb{E}[\mathbb{E}[X_t|\mathcal{F}_0]] = \mathbb{E}[X_t]$$

Applying the supermartingale property once more, as $T > t$,

$$\mathbb{E}[X_t] \geq \mathbb{E}[\mathbb{E}[X_T|\mathcal{F}_t]] = \mathbb{E}[X_T]$$

Combining the two yields:

$$\mathbb{E}[X_0] \geq \mathbb{E}[X_t] \geq \mathbb{E}[X_T]$$

But since by assumption X_t satisfies $\mathbb{E}[X_T] = X_0$, the above equation holds with equality. Therefore for any $t \in [0, T]$, $\mathbb{E}[X_t] = X_0$.

Now consider the random variable defined by:

$$Y := X_t - \mathbb{E}\left[X_T|\mathcal{F}_t\right] \geq 0 \text{ a.s.}$$

Taking expectations of both sides, we have shown above the RHS is zero:

$$\mathbb{E}\left[Y\right] = \mathbb{E}\left[X_t\right] - \mathbb{E}\left[X_T\right] = 0$$

Therefore, $Y = 0$ a.s. and

$$\mathbb{E}\left[X_T|\mathcal{F}_t\right] = X_t$$

Since we assume $\mathbb{E}\left[X_T\right] = X_0 < \infty$, then by the martingale construction Lemma X_t is a martingale. ■

Definition 1.2.5: Local Martingales

A process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)$ is a *local martingale* on $[0, T]$ if there exists a sequence of stopping times $(\tau_n)_{n=1}^{\infty}$ s.t.

(a) $\tau_n \uparrow T \quad \mathbb{P}$-a.s.

(b) $X_{\tau_n \wedge t}$ i s a martingale for any n

where $\tau_n \wedge t := \min\{\tau_n, t\}$.

Lemma 1.2.6

A RCLL local martingale bounded from below is a supermartingale.

Proof for Lemma

(Solution in Chapter 3). By the definition of a local martingale, there exists a sequence of stopping times $\tau_n \uparrow T$ s.t.

$$M_t^n := X_{t \wedge \tau_n} \geq 0$$

is a martingale for any n. For $s \leq t$, the martingale property is

$$\mathbb{E}\left[M_t^n|\mathcal{F}_s\right] = M_s^n$$

Next we apply Fatou's Lemma as $M_t^n \geq 0$ for all $t \in [0, T]$

$$\mathbb{E}\left[\liminf_{n\to\infty} M_t^n|\mathcal{F}_s\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[M_t^n|\mathcal{F}_s\right]$$

Note that as $n \to \infty$, $M_t^n = X_t$ which means the LHS converges to $\mathbb{E}\left[X_t|\mathcal{F}_s\right]$. By the same logic, the RHS is equal to X_s as shown below on the right

$$\begin{aligned} \mathbb{E}\left[X_t|\mathcal{F}_s\right] &\leq \liminf_{n\to\infty} \mathbb{E}\left[M_t^n|\mathcal{F}_s\right] \\ &= \liminf_{n\to\infty} M_s^n \\ &= X_s \end{aligned}$$

Therefore X_t is a supermartingale. ■

Theorem 1.2.7: Doob's optimal sampling theorem

Suppose M is a (RCLL) martingale w.r.t. $(\mathcal{F}_t)$. Let $\nu \leq \tau$ be two bounded stopping times for $(\mathcal{F}_t)$, then

$$M_\nu := \mathbb{E}\left[M_\tau | \mathcal{F}_\nu\right]$$

Proof. Let M be a RCLL martingale and $\tau_n \uparrow T$ be a sequence of stopping times. For $t \geq s$, and $A \in \mathcal{F}_s$ we claim $X_t^n := M_{t\wedge\tau_n}$ is a martingale. Indeed,

$$\begin{aligned}
\mathbb{E}\left[X_t^n \mathbf{1}_{\{A\}}\right] &= \mathbb{E}\left[M_{t\wedge\tau_n}\mathbf{1}_{\{A\}}\right] \\
&= \mathbb{E}\left[M_{t\wedge\tau_n}\mathbf{1}_{\{\tau_n \leq s\}}\mathbf{1}_{\{A\}} + M_{t\wedge\tau_n}\mathbf{1}_{\{\tau_n \leq s\}}\mathbf{1}_{\{A\}}\right] \\
&= \mathbb{E}\left[M_{\tau_n}\mathbf{1}_{\{\tau_n \leq s\}}\mathbf{1}_{\{A\}}\right] + \mathbb{E}\left[M_{t\wedge\tau_n}\mathbf{1}_{\{\tau_n > s\}}\mathbf{1}_{\{A\}}\right] \\
&= \mathbb{E}\left[M_{\tau_n}\mathbf{1}_{\{\tau_n \leq s\}}\mathbf{1}_{\{A\}}\right] + \mathbb{E}\left[\mathbb{E}\left[M_{t\wedge\tau_n}\mathbf{1}_{\{\tau_n > s\mathbf{1}_{\{A\}}\}}|\mathcal{F}_{s\wedge\tau_n}\right]\right] \\
&= \mathbb{E}\left[M_{\tau_n}\mathbf{1}_{\{\tau_n \leq s\}}\mathbf{1}_{\{A\}}\right] + \mathbb{E}\left[\mathbb{E}\left[M_{t\wedge\tau_n}|\mathcal{F}_{s\wedge\tau_n}\right]\mathbf{1}_{\{\tau_n > s\mathbf{1}_{\{A\}}\}}\right]
\end{aligned}$$

□

Corollary 1.2.8

RCLL martingales are local martingales.

Proof for Corollary.

Application of Doob's optional sampling theorem.

1.3 Markov processes

Definition 1.3.1: Markov

A process X on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is *Markov* if, for any time t and any integrable random variable Y that is measurable w.r.t. the σ-algebra generated by $\{X_s : s \geq t\}$, we have:

$$\mathbb{E}\left[Y | \mathcal{F}_t\right] = \mathbb{E}\left[Y | X_t\right]$$

If $Y = f(X_s)$ with $s \geq t$, then f is measurable with finite expectation.

Intuitively, at time t the value X_t tells us all we need to know to compute conditional expectations. The rest is irrelevant. Note Markov property is dependent on filtration $(\mathcal{F}_t)$ and probability $\mathbb{P}$.

Theorem 1.3.2

Let X be a Markov process w.r.t. the filtration $(\mathcal{F}_t)$ and f be a Borel function. Then for a given $s \geq t \geq 0$, there is a Borel-measurable function g s.t.

$$\mathbb{E}\left[f(X_s)|\mathcal{F}_t\right] = \mathbb{E}\left[f(X_s)|X_t\right] = g(X_t)$$

1.4 Brownian motion

Definition 1.4.1: Brownian motion

A continuous time stochastic process W with values in $\mathbb{R}^d$ is a *Brownian motion* if:

1. $\mathbb{P}\left(W_0 = 0\right) = 1$
2. for any $0 < t < s$,
$$W_s - W_t \sim N\left(\vec{0}, (s-t)\mathbf{I}_d\right)$$
3. for all times $t_0, \ldots, t_n$ with $0 < t_1 < \cdots < t_n < \infty$, the random variables
$$W_{t_0},\ W_{t_1} - W_{t_2}, \ldots,\ W_{t_n} - W_{t_{n-1}}$$
are independent.

Definition 1.4.2: Brownian motion w.r.t. a filtration

We say that a stochastic process W on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with values in $\mathbb{R}^d$ is a Brownian motion w.r.t. the filtration $\mathcal{F}_t$ if:

1. it is a Brownian motion
2. for each $t > 0$, W_t is $\mathcal{F}_t$-measurable.
3. for $0 \leq t < s$ the increment $W_s - W_t$ is independent of $\mathcal{F}_t$.

Remark.

> A Brownian motion w.r.t. a smaller filtration might fail to be such with respect to a larger one (insider trading). On the other hand, if a Brownian motion w.r.t.larger filtration is adapted to a smaller one, it would be a Brownian motion under it.
>
> The filtration under which the two above definitions coincide is the **natural filtration**:
>
> $$\mathcal{F}_t^W = \sigma(W_s, s \leq t)$$
>
> Note this filtration is only right continuous if we add the null set, i.e. $F_t := \sigma(\mathcal{F}_t^W \cup \mathcal{N})$.

Property 1.4.3: Properties of Brownian motion

Brownian motion is a Gaussian process with:

(i) $\mathbb{E}\left[W_t\right] = 0$

(ii) $\operatorname{cov}\left(W_s, W_t\right) = \min\{t, s\}$

Proof

> (i) For any $t \geq 0$,
> $$\mathbb{E}\left[W_t\right] = \mathbb{E}\left[W_t\right] - \underbrace{\mathbb{E}\left[W_t\right]}_{=0} = \mathbb{E}\left[W_t - W_0\right] = \vec{0}$$

(ii) Assume $t \geq s$, then

$$\begin{aligned} \operatorname{cov}(W_t, W_s) &= \mathbb{E}[W_t W_s] \\ &= \mathbb{E}[((W_t - W_s) + W_s)W_s] \\ &= \mathbb{E}[(W_t - W_s)W_s] + \mathbb{E}\left[W_s^2\right] \\ &= 0 + s \end{aligned}$$

Lemma 1.4.4: Brownian motion is Markov

Let W_t be a Brownian motion w.r.t. the filtration $(\mathcal{F}_t)$. Then W_t is a Markov process.

Proof for Lemma

Lemma 1.4.5: Martingale property of Brownian motion

Brownian motion is a *martingale*.

Proof for Lemma

By Jensen's inequality: $\mathbb{E}[|W_t|] \leq \mathbb{E}\left[W_t^2\right]^{1/2} = \sqrt{t} < \infty$. Thus for $t > s$,

$$\begin{aligned} \mathbb{E}[W_t|\mathcal{F}_s] &= \mathbb{E}[W_t - W_s + W_s|\mathcal{F}_s] \\ &= W_s + \mathbb{E}[W_t - W_s|\mathcal{F}_s] \\ &= W_s + \mathbb{E}[W_t - W_s] \\ &= W_s \end{aligned}$$

Definition 1.4.6: H-self-similar

A process X is H-*self-similar* for some $H > 0$ if

$$(T^H X_{t_1}, \ldots, T^H X_{t_n}) \overset{d}{=} (X_{Tt_1}, \ldots, X_{Tt_n})$$

for any $T > 0$, and any choice of $t_i \geq 0, i = 1, \ldots, n; n \geq 1$.

Property 1.4.7: Fractal property of Brownian motion

Brownian motion is $1/2$-self-similar, that is magnifying BM will not make its paths any less jagged.

This is a great building block as it is scale independent.

1.4.1 Variation

Definition 1.4.8: Variation

The p-th variation of a continuous function $f : [0,T] \to \mathbb{R}$ on an interval $[0,T]$ is defined as:

$$V_T^{(p)}(f) := \lim_{\|\pi^n\| \to 0} \sum_{i=0}^{n-1} \left| f(t_{i+1}^n) - f(t_i^n) \right|^p$$

where π^n is a partition of interval $[0,T]$, i.e.

$$0 = t_0^n < t_1^n < \cdots < t_n^n = T$$

and $\|\pi^n\| := \max_i \left| t_{i+1}^n - t_i^n \right|$.

If $p = 1$, then $V_T^{(1)}$ represents the length of f travelled on $[0,T]$.

Remark.

Observe that if $V_T^{(1)}(f) < \infty$, then $V^{(2)}(f) = 0$ as

$$\begin{aligned} V_T^{(2)} &:= \lim_{\|\pi^n\| \to 0} \sum_{i=0}^{n-1} \left| f(t_{i+1}^n) - f(t_i^n) \right|^2 \\ &\leq \lim_{\|\pi^n\| \to 0} \max_i \{ \left| f(t_{i+1}^n) - f(t_i^n) \right| \} V_T^{(1)}(f) \end{aligned}$$

Lemma 1.4.9: Unbounded variation of Brownian motion

Brownian motion travels an infinite distance between times $[s,t]$. We say it is of *unbounded variation*.

Proof for Lemma

For a Brownian motion W we have:

$$\mathbb{E}\left[\sum_{i=0}^{n-1} \left| W_{t_{i+1}^n} - W_{t_i^n}^2 \right| \right] =$$

1.4.2 Arithmetic and Geometric Brownian motion

Definition 1.4.10: Arithmetic Brownian motion

A process X is called an *arithmetic Brownian motion* or *Brownian motion with drift* if

$$X_t = \mu t + \sigma W_t; \quad t \geq 0$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. This process is a generalisation of a standard Brownian motion.

Property 1.4.11: Properties of ABM

It holds that:

1. $\mathbb{E}\left[X_t\right] = \mu t$
2. $\operatorname{cov}\left(X_s, X_t\right) = \sigma^2 \min\{t, s\}$

Proof

Trivial

Definition 1.4.12: Geometric Brownian motion

A process X is called an *geometric Brownian motion* if

$$X_t = e^{\mu t + \sigma W_t}; \quad t \geq 0$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

Property 1.4.13: Properties of GBM

It holds that:

(i) $\mathbb{E}\left[X_t\right] = e^{\left(\mu + \frac{1}{2}\sigma^2\right)t}$

(ii) $\operatorname{var}\left(X_t\right) = e^{(2\mu + \sigma^2)t}\left(e^{\sigma^2 t} - 1\right)$

Proof

tbc

Example.

The following GBM is a **martingale**:

$$X_t := \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right)$$

Proof. $\square$

Example.

(Local martingale that is not a martingale) Let W be a Brownian motion w.r.t. a filtered space. Then

$$\tau := \inf\{t \geq 0 : W_t \geq 1\}$$

is a stopping time.

1.5 Exercises

Ex.1. Let W_t be a standard one-dimensional Brownian motion. Given times $r < s < t < u$, calculate the expectations

(i) $\mathbb{E}\left[(W_t - W_s)(W_s - W_r)\right]$

(ii) $\mathbb{E}\left[(W_u - W_t)^2(W_s - W_r)^2\right]$

(iii) $\mathbb{E}\left[(W_u - W_s)(W_t - W_r)\right]$

(iv) $\mathbb{E}\left[(W_t - W_r)(W_t - W_r)^2\right]$

(v) $\mathbb{E}\left[W_r W_s W_t\right]$

Ex.2. Let W_t be a standard Brownian motion. Given a constant $c > 0$, show that the stochastic process X_t defined by:

$$X_t = \frac{1}{\sqrt{c}} W_{c,t}$$

is a standard Brownian motion.

Ex.3. Suppose that the process W_t is a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion.

(i) Prove that the process X_t defined by

$$X_t = W_t^2 - t; \quad \text{for } t \geq 0$$

is an $(\mathcal{F}_t)$-martingale.

(ii) Prove the process Y_t defined by

$$Y_t = \exp\left(-\frac{1}{2}\theta^2 t - \theta W_t\right); \quad \text{for } t \geq 0$$

is an $(\mathcal{F}_t)$-martingale.

1.6 Solutions

Ex.1. For

(i) We have:

$$\begin{aligned}\mathbb{E}\left[(W_t - W_s)(W_s - W_r)\right] &= \mathbb{E}\left[\mathbb{E}\left[(W_t - W_s)(W_s - W_r)|\mathcal{F}_s\right]\right]\\ &= \mathbb{E}\left[(W_s - W_r)\mathbb{E}\left[W_t - W_s|\mathcal{F}_s\right]\right]\\ &= \mathbb{E}\left[(W_s - W_r)\mathbb{E}\left[W_t - W_s\right]\right]\\ &= \mathbb{E}\left[(W_s - W_r)\cdot 0\right]\\ &= 0\end{aligned}$$

(ii)

(iii)

(iv) We have:

$$\begin{aligned}\mathbb{E}\left[(W_t - W_r)(W_t - W_r)^2\right] &= \mathbb{E}\left[(W_t - W_s + W_s - W_r)(W_s - W_r)^2\right]\\ &= \mathbb{E}\left[(W_t - W_s)(W_s - W_r)^2 + (W_s - W_r)^3\right]\\ &= \mathbb{E}\left[W_t - W_s\right]\mathbb{E}\left[(W_s - W_r)^2\right] + \mathbb{E}\left[(W_s - W_r)^3\right]\\ &= 0\end{aligned}$$

where the last line follows as the third normal moment is zero.

Ex.2. We want to show the properties of Brownian motion, which are

(a) $X_0 = 0$

(b) for $s > t > 0$, $X_s - X_t \sim N(0, s-t)$

(c) independence of non-overlapping intervals

We proceed; for:

(a) We have that $W_0 = 0 \iff c^{-1/2}W_{c\cdot 0} = 0 \iff X_0 = 0$

(b) Note X_t is mean zero as $s > t > 0 \iff cs > ct > 0$:

$$\begin{aligned}\mathbb{E}\left[X_s - X_t\right] &= c^{-1/2}\mathbb{E}\left[W_{cs} - W_{ct}\right]\\ &= c^{-1/2}\cdot 0\\ &= 0\end{aligned}$$

and has variance $s - t$

$$\begin{aligned}\mathrm{var}\left(X_s - X_t\right) &= c^{-1}\mathrm{var}\left(W_{cs} - W_{ct}\right)\\ &= c^{-1}(cs - ct)\\ &= s - t\end{aligned}$$

It is normally distributed as it is a linear combination of a normally distributed variable.

(c) Covariances of non-overlapping intervals are zero is sufficient for independence when normal. As $0 < t_1 < \cdots < t_n < \infty \implies 0 < ct_1 < \cdots < ct_n < \infty$, thus the independence of $W_{ct_n} - W_{ct_{n-1}}$ implies $X_{t_n} - X_{t_{n-1}}$ are independent.

Ex.3. For each part we want to show the variable satisfies martingale properties, which recall were:

1. X is $(\mathcal{F}_t)$-adapted
2. $\mathbb{E}\left[|X_t|\right] < \infty$
3. for all $s > t$, $\mathbb{E}\left[X_s|\mathcal{F}_t\right] = X_t$

(i) We check the conditions in order:

1. The function $f(x_t) = x_t^2 - t$ is non-stochastic and continuous thus measurable and adapted.
2. Apply triangle inequality:
$$|X_t| \leq \left|W_t^2\right| + |-t| = 2t < \infty$$
3. Note for $s \geq t$
$$\begin{aligned}
\mathbb{E}\left[X_s|\mathcal{F}_t\right] &= \mathbb{E}\left[W_s^2 - s|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[W_s - W_t + W_t)^2|\mathcal{F}_t\right] - s \\
&= \mathbb{E}\left[(W_s - W_t)^2 + 2(W_s - W_t)W_t + W_t^2|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[(W_s - W_t)^2|\mathcal{F}_t\right] + \mathbb{E}\left[2(W_s - W_t)W_t|\mathcal{F}_t\right] + W_t^2 - s \\
&= \mathbb{E}\left[(W_s - W_t)^2\right] + 2W_t\mathbb{E}\left[W_s - W_t|\mathcal{F}_t\right] + W_t^2 - s \\
&= s - t + W_t^2 - s \\
&= W_t^2 - t \\
&= X_t
\end{aligned}$$

Thus X_t satisfies all three martingale conditions.

(ii) We check the conditions in order:

1. The function is non-stochastic and continuous thus measurable and adapted.
2. The exponential function is positive, thus:
$$\mathbb{E}\left[|Y_t|\right] = \mathbb{E}\left[Y_t\right] = e^{-\frac{1}{2}\theta^2 t}\mathbb{E}\left[e^{-\theta W_t}\right]$$
The $MGF(n)$ of normally distributed variable, Z is $\mathbb{E}\left[e^{nZ}\right] = e^{\mu n + \frac{1}{2}\sigma^2 n^2}$, therefore taking "$\theta$" moments, we have:
$$\mathbb{E}\left[|Y_t|\right] = e^{-\frac{1}{2}\theta^2 t}e^{\frac{1}{2}\theta^2 t} = 1 < \infty$$
3. For the martingale property we have for $s \geq t$
$$\begin{aligned}
\mathbb{E}\left[Y_s|\mathcal{F}_t\right] &= \mathbb{E}\left[\exp\left(-\frac{1}{2}\theta^2 s - \theta(W_s - W_t) - \theta W_t\right)\middle|\mathcal{F}_t\right] \\
&= \exp\left(-\frac{1}{2}\theta^2 s - \theta W_t\right)\mathbb{E}\left[\exp\left(-\theta(W_s - W_t)\right)\right] \\
&= \exp\left(-\frac{1}{2}\theta^2 s - \theta W_t\right)\exp\left(\frac{1}{2}\theta^2(s - t)\right) \\
&= \exp\left(-\frac{1}{2}\theta^2 t - \theta W_t\right) \\
&= Y_t
\end{aligned}$$
where the second equality follows from the independence of $W_s - W_t$ w.r.t. $\mathcal{F}_t$ and the measurability of W_t w.r.t. $\mathcal{F}_t$.

Chapter 2

Itô Calculus

2.1 Itô integral

2.1.1 Motivation

Suppose we have the standard Brownian motion W as a model for price. If we bought π_0 assets at time 0 and hold them until time T, the value of out portfolio at time T will be:

$$\pi_0(W_T - W_0)$$

Now suppose we decided at deterministic times $t_0, t_1, \ldots, t_n$ we want $\pi_0, \pi_1, \ldots, \pi_n$ assets respectively. Then at T we will have:

$$\sum_{i=0}^{n} \pi_i (W_{t_{i+1}} - W_{t_i})$$

Note the above holds even if π_i is random. What is more interesting is what happens if we take $t_{i+1} - t_i$ smaller and smaller.

Example.

> **A cautionary example**: based on standard calculus, one may guess the answer to the integral below to be:
>
> $$\int_0^T W_t \, \mathrm{d}W_t \overset{?}{=} \frac{1}{2} W_T^2$$
>
> However, if we approximate the rectangle under the curve with the left limit or right limit will change the value of the integral. If we take **left approximations**, then we have:
>
> $$\begin{aligned} \sum_{i=1}^{n} W_{t_i}(W_{t_{i+1}} - W_{t_i}) &= \sum_{i=1}^{n} W_{t_i} W_{t_{i+1}} - W_{t_i}^2 \\ &= \sum_{i=1}^{n} \frac{1}{2}(W_{t_{i+1}}^2 + W_{t_i}^2 - (W_{t_{i+1}}^2 - W_{t_i}^2)^2) - W_{t_i}^2 \\ &= \frac{1}{2} \underbrace{\sum_{i=1}^{n} W_{t_{i+1}}^2 - W_{t_i}^2}_{\text{telescopic sum}} - \frac{1}{2} \sum_{i=1}^{n} (W_{t_{i+1}} - W_{t_i})^2 \\ &= \frac{1}{2} \sum_{i=1}^{n} W_T^2 - \underbrace{\frac{1}{2} \sum_{i=1}^{n} (W_{t_{i+1}} - W_{t_i})^2}_{\text{quad. var.}=T} \end{aligned}$$

$$= \frac{W_T^2 - T}{2}$$

where the second line comes from the identity $W_{t_{i+1}} W_{t_i} \equiv \frac{1}{2}(W_{t_{i+1}}^2 + W_{t_i}^2 - (W_{t_{i+1}}^2 - W_{t_i}^2)^2)$ and the last comes from the previous chapter on quadratic variation. On the other hand if we take the **right approximation**, it can be verified that we get:

$$\sum_{i=1}^{n} W_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) = \frac{W_T^2 + T}{2}$$

However these two 'integrals' are different due to **finite quadratic variation**. Neither integral are 'wrong'; the Itô integral

2.1.2 Itô integral for simple processes

Definition 2.1.1: Simple process

A process X on $[0,T]$ is said to be a *simple process* if there exists a partition

$$0 = t_0 < t_1 < \cdots < t_n = T$$

of $[0,T]$ and a sequence of random variables K_i s.t. for $t \in [t_i, t_{i+1})$:

$$X_t = K_i$$

(i.e. a random step function)

The simple process remains constant over time intervals of the form $[t_i, t_{i+1})$. The value of the process over each interval is known at the left end point of the interval.

Definition 2.1.2: Itô integral of a simple process

The *Itô integral*, I_T for a simple process X on $[0,T]$ is defined pathwise as:

$$\int_0^T X_s \,\mathrm{d}W_s := \sum_{i=1}^{n} X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})$$

For a fixed t, I_t is a random variable. The parameterised collection $(I_t)_{t\in[0,T]}$ is a stochastic process.

Note this is just the Riemann-Stieltjes sum of X against W using the **left-end points** of the partitioning time intervals in the intermediate partition. Also note we assumed a filtration $(\mathcal{F}_t)$ s.t. X is $(\mathcal{F}_t)$-adapted and W is a Brownian motion on the filtered space.

Property 2.1.3: Elementary properties of simple Itô integral

The simple Itô integral, $I_t(X) := \int_0^t X_s \,\mathrm{d}W_s$ for a simple process, X_t has the following properties:

(a) **linearity**; $I_t(aX + bY) = aI_t(X) + bI_t(Y)$

(b) **measurability**; $I_t(X)$ is $(\mathcal{F}_t)$-adapted

(c) **continuity**; $t \to I_t(X)$

Proof

For:

(a) summation is linear.

(b)

(c)

Property 2.1.4: Martingale property

The simple Itô integral, $I_t(X) := \int_0^t X_s \, \mathrm{d}W_s$ for a simple process, X_t is an $\mathcal{L}^2$**-martingale**; that is for $T > s$, $\mathbb{E}\left[I_T | \mathcal{F}_s\right] = I_s$. In addition, if $I_0 = 0$ then $\mathbb{E}\left[I_t\right] = 0$.

Proof

By linearity, we have

$$\mathbb{E}\left[I_T | \mathcal{F}_s\right] = \mathbb{E}\left[\sum_{i=1}^n X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) \middle| \mathcal{F}_s\right] = \sum_{i=1}^n \mathbb{E}\left[X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) \middle| \mathcal{F}_s\right]$$

If $t_i < s$, then

$$\mathbb{E}\left[X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) \middle| \mathcal{F}_s\right] = X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})$$

and suppose $t_i > s > t_{i-1}$ then

$$\begin{aligned} \mathbb{E}\left[X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) \middle| \mathcal{F}_s\right] &= X_{t_{i-1}}(\mathbb{E}\left[W_{t_i} | \mathcal{F}_s\right] - W_{t_{i-1}}) \\ &= X_{t_{i-1}}(W_s - W_{t_{i-1}}) \end{aligned}$$

Therefore we can add all the t up to s without problem, all that remains to show is the rest of the sum from s to T is zero;

$$\mathbb{E}\left[I_T | \mathcal{F}_s\right] = I_s + \sum_{i:t_{i-1} > s} \mathbb{E}\left[X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) | \mathcal{F}_s\right]$$

Focusing on the last term,

$$\begin{aligned} \mathbb{E}\left[X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) | \mathcal{F}_s\right] &= \mathbb{E}\left[\mathbb{E}\left[X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) | \mathcal{F}_{t_{i-1}}\right] | \mathcal{F}_s\right] \\ &= \mathbb{E}\left[X_{t_{i-1}}(\mathbb{E}\left[W_{t_i} | \mathcal{F}_{t_{i-1}}\right] - W_{t_{i-1}}) | \mathcal{F}_s\right] \\ &= \mathbb{E}\left[X_{t_{i-1}}(W_{t_{i-1}} - W_{t_{i-1}}) | \mathcal{F}_s\right] = 0 \end{aligned}$$

Therefore $\mathbb{E}\left[I_T | \mathcal{F}_s\right] = I_s$, the martingale property.

Property 2.1.5: Itô Isometry

The simple Itô integral, $I_t(X) := \int_0^t X_s \, \mathrm{d}W_s$ for a simple process, X_t satisfies the *Itô isometry* property

$$\mathbb{E}\left[I_t^2\right] := \mathbb{E}\left[\int_0^t X_u^2 \, \mathrm{d}u\right] = \int_0^t \mathbb{E}\left[X_u^2\right] \mathrm{d}u$$

Proof

We have

$$\mathbb{E}\left[I_T^2\right] = \mathbb{E}\left[\left(\sum_{i=1}^{n} X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})\right)\right]$$

Using the identity, $\left(\sum_{i=1}^n a_i^2\right) = \sum_{i=1}^n a_i^2 + 2\sum_{i=1}^n \sum_{j=i+1}^n a_i a_j$, and setting $a_i = X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})$, we have

$$\mathbb{E}\left[I_T^2\right] = \mathbb{E}\left[\sum_{i=1}^{n} X_{t_{i-1}}^2(W_{t_i} - W_{t_{i-1}})^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})X_{t_{j-1}}(W_{t_j} - W_{t_{j-1}})\right]$$

- Looking at the first term, we have

$$\begin{aligned}\sum_{i=1}^{n}\mathbb{E}\left[X_{t_{i-1}}^2(W_{t_i} - W_{t_{i-1}})^2\right] &= \sum_{i=1}^{n}\mathbb{E}\left[\mathbb{E}\left[X_{t_{i-1}}^2(W_{t_i} - W_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}\right]\right]\\ &= \sum_{i=1}^{n}\mathbb{E}\left[X_{t_{i-1}}^2\mathbb{E}\left[(W_{t_i} - W_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}\right]\right]\\ &= \sum_{i=1}^{n}\mathbb{E}\left[X_{t_{i-1}}^2(t_i - t_{i-1})\right]\\ &= \mathbb{E}\left[\int_0^T X_u^2 \,\mathrm{d}u\right]\end{aligned}$$

  as $\mathbb{E}\left[(W_{t_i} - W_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}\right] = \mathbb{E}\left[(W_{t_i} - W_{t_{i-1}})^2\right] = t_i - t_{i-1}$.

- The second term is zero by using LIE conditioning on $\mathcal{F}_{t_{i-1}}$. Since the Brownian motion is a martingale, this leaves zero.

Lemma 2.1.6: Quadratic variation of Itô simple integral

Quadratic variation of Itô integral is given by:

$$\langle I_t \rangle_t = \int_0^t X_u^2 \,\mathrm{d}u$$

Proof for Lemma

2.1.3 Itô integral for Brownian motion

We can now define $\int_0^t \theta_s \,\mathrm{d}W_s$ for any process in $\mathcal{H}^2$, i.e. any process that satisfies the **regularity condition**:

$$\mathbb{E}\left[\int_0^t \theta_s^2 \,\mathrm{d}s\right] < \infty$$

Proceed as follows:

Step 1. Approximate a process θ in $\mathcal{H}^2$ by a sequence of simple processes; there exists a sequence of simple processes (X^n) such that:

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^t [X_s^n - \theta_s]^2 \,\mathrm{d}s\right] = 0$$

That is, (X^n) converges to θ in the $\mathcal{L}^2(\mathbb{P}\times\lambda)$ norm.

Step 2. Since X^n_s converges to θ_s, then it is Cauchy. Let $I^n_t := \int_0^t X^n_s \,\mathrm{d}W_s$ i.e. the well defined integral of the n-th process in the sequence of simple approximating processes. For $n, m \in \mathbb{N}$, since X^n_t is Cauchy the LHS converges to 0, and applying Itô isometry to the second line, we show that I^n_t is Cauchy

$$\begin{aligned}
\|X^n_s - X^m_s\|_{\mathcal{L}^2(\mathbb{P}\times\lambda)} &= \mathbb{E}\left[\int_0^t [X^n_s - X^m_s]^2 \,\mathrm{d}s\right] \\
&= \mathbb{E}\left[\left[\int_0^t X^n_s \,\mathrm{d}s - \int_0^t X^m_s \,\mathrm{d}s\right]^2\right] \\
&= \mathbb{E}\left[(I^n_t - I^m_t)^2\right] \\
&= \|I^n_t - I^m_t\|_{\mathcal{L}^2(\mathbb{P})} \to 0
\end{aligned}$$

Since I^n_t is Cauchy, it is convergent on $\mathcal{L}^2$ as it can be proven $\mathcal{L}^2$ is a complete space. The limit is defined as the *Itô integral*.

Definition 2.1.7: W-integrable

The

Definition 2.1.8: W-local-integrable

The

Property 2.1.9: Elementary properties of Itô integral

The Itô integral, $I_t(X) := \int_0^t X_s \,\mathrm{d}W_s$ for stochastic process, $X_t \in \mathcal{H}^2_{\text{loc}}$ has the following properties:

(a) **linearity**; $I_t(aX + bY) = aI_t(X) + bI_t(Y)$

(b) **measurability**; $I_t(X)$ is $(\mathcal{F}_t)$-adapted

(c) **continuity**; $t \to I_t(X)$

Proof

For:

(a) summation is linear.

(b)

(c)

Property 2.1.10: Martingale property

The Itô integral, $I_t(X) := \int_0^t X_s \, \mathrm{d}W_s$ for stochastic process, $X_t \in \mathcal{H}^2$ is an $\mathcal{L}^2$-**martingale**; that is for $T > s$,

$$\mathbb{E}\left[I_T | \mathcal{F}_s\right] = I_s$$

In addition, if $I_0 = 0$ then $\mathbb{E}\left[I_t\right] = 0$.

Proof

For $\theta \in \mathcal{H}^2$, and the approximating integral I^n that satisfies $\lim_{n\to\infty} \mathbb{E}\left[I_T^n - I_T\right] = 0$ we have

$$\begin{aligned} 0 \leq \mathbb{E}\left[\left(\mathbb{E}\left[I_T|\mathcal{F}_s\right] - I_s\right)^2\right] &= \mathbb{E}\left[\left(\mathbb{E}\left[I_T - I_T^n|\mathcal{F}_s\right] - (I_s - I_s^n)\right)^2\right] \\ &= \mathbb{E}\left[\left(\mathbb{E}\left[I_T - I_T^n|\mathcal{F}_s\right]\right)^2\right] + 2\mathbb{E}\left[\mathbb{E}\left[I_T - I_T^n|\mathcal{F}_s\right](I_s^n - I_s)\right] + \mathbb{E}\left[(I_s - I_s^n)^2\right] \\ &\leq 2\mathbb{E}\left[\left(\mathbb{E}\left[I_T - I_T^n|\mathcal{F}_s\right]\right)^2\right] + 2\,\underbrace{\mathbb{E}\left[(I_s - I_s^n)^2\right]}_{\to 0} \end{aligned}$$

where the third line is obtained using the identity $2ab \leq a^2 + b^2$. By Jensen's (conditional) inequality, the first term of the last line satisfies

$$\mathbb{E}\left[\left(\mathbb{E}\left[I_T - I_T^n|\mathcal{F}_s\right]\right)^2\right] \leq \mathbb{E}\left[\mathbb{E}\left[(I_T - I_T^n)^2|\mathcal{F}_s\right]\right] = \mathbb{E}\left[(I_T - I_T^n)^2\right] \to 0$$

Thus

$$\mathbb{E}\left[\left(\mathbb{E}\left[I_T|\mathcal{F}_s\right] - I_s\right)^2\right] = 0 \implies \left(\mathbb{E}\left[I_T|\mathcal{F}_s\right] - I_s\right)^2 = 0 \implies \mathbb{E}\left[I_T|\mathcal{F}_s\right] = I_s$$

Property 2.1.11: Itô Isometry

The Itô integral, $I_t(X) := \int_0^t X_s \, \mathrm{d}W_s$ for a stochastic process, $X_t \in \mathcal{H}^2$ satisfies the *Itô isometry* property

$$\mathbb{E}\left[I_t^2\right] := \mathbb{E}\left[\int_0^t X_u^2 \, \mathrm{d}u\right] = \int_0^t \mathbb{E}\left[X_u^2\right] \mathrm{d}u$$

Proof

Lemma 2.1.12: Quadratic variation of Itô simple integral

Quadratic variation of Itô integral is given by:

$$\langle I_t \rangle_t = \int_0^t X_u^2 \, \mathrm{d}u$$

Proof for Lemma

Example.

> An example of computing the Itô integral:
>
> $$\int_0^T W_t \,\mathrm{d}W_t = \frac{W_T^2}{2} - \frac{T}{2}$$

Proof. Guess the simple process:

$$\theta_t^n := W_{t_i^n} \mathbf{1}_{[t_i^n, t_{i+1}^n)}$$

Check it approximates W_t in $\mathcal{H}^2$, that is we want

$$\mathbb{E}\left[\int_0^T (\theta_t^n - W_t)^2 \,\mathrm{d}t\right] \to 0$$

We have:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T (\theta_t^n - W_t)^2 \,\mathrm{d}t\right] &\overset{\text{Isom.}}{=} \int_0^T \mathbb{E}\left[(\theta_t^n - W_t)^2\right] \mathrm{d}t \\
&= \int_0^T \sum_{i=1}^n \mathbb{E}\left[(W_{t_i^n} - W_t)^2\right] \mathbf{1}_{t \in [t_i^n, t_{i+1}^n)} \,\mathrm{d}t \\
&= \int_0^T \sum_{i=1}^n (t_i^n - t) \mathbf{1}_{t \in [t_i^n, t_{i+1}^n)} \,\mathrm{d}t \to 0
\end{aligned}$$

Now we proceed with this sequence.

$$\begin{aligned}
\mathbb{E}\left[(I_T^n - V)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n W_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n}) - \frac{W_T^2}{2} + \frac{T}{2}\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^n \left(W_{t_i^n} W_{t_{i+1}^n} - W_{t_i}^2\right) - \frac{1}{2}\sum_{i=1}^n (W_{t_{i+1}}^2 - W_{t_i}^2) + \frac{1}{2}\sum_{i=1}^n (t_{i+1} - t_i)\right)^2\right] \\
&= \frac{1}{4}\mathbb{E}\left[\sum_{i=1}^n \left[(t_{i+1}^n - t_i^n - (W_{t_{i+1}^n} - W_{t_i^n})^2\right]^2\right]
\end{aligned}$$

tbc... ☐

2.2 Itô Processes

Definition 2.2.1: Itô Process

We call a stochastic process X an *Itô process* if it can be represented as

$$X_t = X_0 + \int_0^t \mu_s \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}W_s$$

where $X_0 \in \mathbb{R}$, $\sigma \in \mathcal{H}^2$,$\mu \in \mathcal{L}^1$ (i.e. $\int_0^t |\mu_s| \,\mathrm{d}s < \infty$ a.s.).

In **differential form**, we can express the dynamic as:

$$\mathrm{d}X_t = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t$$

The first integral is a time integral, while the second is an Itô integral. Itô processes provide us with one of the most flexible probabilistic models of security prices (without jumps).

Definition 2.2.2: Drift and diffusion

We can interpret the process μ_t as the *drift*, and σ_t as the diffusion process as

$$\begin{aligned}\mathbb{E}_t\left[\mathrm{d}S_t\right] &= \mu_t\,\mathrm{d}t\\ \operatorname{var}_t\left(\mathrm{d}S_t\right) &= \sigma_t^2\,\mathrm{d}t\end{aligned}$$

Lemma 2.2.3: Almost sure, almost everywhere processes

Let

$$\begin{aligned}X_t &= X_0 + \int_0^t a_s\,\mathrm{d}s + \int_0^t b_s\,\mathrm{d}W_s\\ Y_t &= Y_0 + \int_0^t \alpha_s\,\mathrm{d}s + \int_0^t \beta_s\,\mathrm{d}W_s\end{aligned}$$

We have $X_t = Y_t$ a.s. a.e. if and only if $X_0 = Y_0$ and $a = \alpha$, $b = \beta$ a.s. a.e.

Proof for Lemma

Lemma 2.2.4: Quadratic variation of Itô process

The quadratic variation of an Itô process is

$$\langle S\rangle_t = \int_0^t \sigma_s^2\,\mathrm{d}s$$

Proof for Lemma

Let π^n be a sequence of partitions of $[0,t]$ s.t. $\lim_{n\to\infty}\sum_{i=0}^{n-1}\left(\int_{t_i^n}^{t_{i+1}^n}\sigma_s\,\mathrm{d}W_s\right)^2 = \int_0^t \sigma_s^2\,\mathrm{d}s$. Quadratic variation is defined as

$$\begin{aligned}V_t^2(S) &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left(S_{t_{i-1}^n} - S_{t_i^n}\right)^2\\ &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left(\int_{t_i^n}^{t_{i+1}^n}\mu_s\,\mathrm{d}s + \int_{t_i^n}^{t_{i+1}^n}\sigma_s\,\mathrm{d}W_s\right)^2\\ &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left(\int_{t_i^n}^{t_{i+1}^n}\mu_s\,\mathrm{d}s\right)^2 + 2\int_{t_i^n}^{t_{i+1}^n}\mu_s\,\mathrm{d}s\int_{t_i^n}^{t_{i+1}^n}\sigma_s\,\mathrm{d}W_s + \left(\int_{t_i^n}^{t_{i+1}^n}\sigma_s\,\mathrm{d}W_s\right)^2\end{aligned}$$

We deal with each part of the sum separately;

- for

$$\begin{aligned}\sum_{i=0}^{n-1}\left(\int_{t_i^n}^{t_{i+1}^n}\mu_s\,\mathrm{d}s\right)^2 &\le \sup_i\int_{t_i^n}^{t_{i+1}^n}|\mu_s|\,\mathrm{d}s\left(\sum_{i=0}^{k-1}\int_{t_i^n}^{t_{i+1}^n}|\mu_s|\,\mathrm{d}s\right)\\ &= \sup_i\int_{t_i^n}^{t_{i+1}^n}|\mu_s|\,\mathrm{d}s\left(\int_0^t|\mu_s|\,\mathrm{d}s\right)\end{aligned}$$

$$\le \sup_{[0,t]} |\mu_s| \sup_i (t^n_{i+1} - t^n_i) \left(\int_0^t |\mu_s| \,\mathrm{d}s \right) \to 0$$

as $\sup_i (t^n_{i+1} - t^n_i) = \|\pi^n\| \to 0$ and the integral is finite.

- for

$$\sum_{i=0}^{n-1} \int_{t^n_i}^{t^n_{i+1}} \mu_s \,\mathrm{d}s \int_{t^n_i}^{t^n_{i+1}} \sigma_s \,\mathrm{d}W_s \le \int_0^t |\mu_s| \,\mathrm{d}W_s \sup_i \left| \int_{t^n_i}^{t^n_{i+1}} \sigma_s \,\mathrm{d}W_s \right| \to 0$$

again as $\sup_i (t^n_{i+1} - t^n_i) = \|\pi^n\| \to 0$ and the integral is finite.

- the last term converges to quadratic variation of the Itô integral,

$$\left(\int_{t^n_i}^{t^n_{i+1}} \sigma_s \,\mathrm{d}W_s \right)^2 \underset{a.s.}{\to} \int_0^t \sigma_s^2 \,\mathrm{d}s$$

Thus we are left with $V_t^2(S) = \int_0^t \sigma_s^2 \,\mathrm{d}s$ ■

Definition 2.2.5: Itô integral w.r.t. an Itô process

Let θ be an adapted process s.t. $\mu\theta \in \mathcal{L}^2$ and $\theta\sigma \in \mathcal{H}^2$. The integral of θ w.r.t. the Itô process S_t is defined as

$$\int_0^t \theta_u \,\mathrm{d}S_u := \int_0^t \theta_u \mu_u \,\mathrm{d}u + \int_0^t \theta_u \sigma_u \,\mathrm{d}W_u$$

2.3 Itô formula

Recall the Taylor expansion:

Definition 2.3.1: Taylor formula

A Taylor approximation of $f : \mathbb{R} \to \mathbb{R}$ at t is given by:

$$f(t + \Delta) = f(t) + f'(t)\Delta + \frac{1}{2} f''(t)\Delta^2 + \cdots + \frac{1}{n!} f^{(n)}(t)\Delta^n$$

Definition 2.3.2: Order symbol

For a random sequence X_n,

$$X_n = \bar{o}(a_n) \iff \lim_{n\to\infty} \frac{X_n}{a_n} \xrightarrow{p} 0$$

Example.

We show the Fundamental Theorem of Calculus with the Taylor formula. Partition $[0, T]$ with $\Delta := T/n$.

$$f(T) - f(0) = \lim_{n\to\infty} \left\{ \left[f\left(\tfrac{T}{n}\right) - f(0) \right] + \left[f\left(\tfrac{2T}{n}\right) - f\left(\tfrac{T}{n}\right) \right] + \cdots \left[f(T) - f\left(\tfrac{(n-1)T}{n}\right) \right] \right\}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\tfrac{(i-1)T}{n} + \tfrac{T}{n}\right) - f\left(\tfrac{(i-1)T}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[f'\left(\tfrac{(i-1)T}{n}\right)\tfrac{T}{n} + \bar{o}\left(\tfrac{1}{n}\right)\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[f'\left(\tfrac{(i-1)T}{n}\right)\tfrac{T}{n}\right] + \bar{n}o\left(\tfrac{1}{n}\right) \\
&= \int_0^t f'(u)\,\mathrm{d}u
\end{aligned}
$$

where the third equality follows by Taylor's formula above.

Now suppose we have S_t instead; a deterministic smooth function, and we want to apply Taylor's formula to $f(S_t)$.

$$f(S_{t+\Delta}) = f\left(S_t + S_t'\Delta + \bar{o}(\tilde{\Delta})\right)$$

Let $\tilde{\Delta} := (S_t'\Delta + \bar{o}(\Delta))$, then we also have that

$$
\begin{aligned}
=& f(S_t) + f'(S_t)\tilde{\Delta} + \bar{o}(\Delta) \\
=& f(S_t) + f'(S_t)(S_t'\Delta + \bar{o}(\Delta)) + \bar{o}(\tilde{\Delta}) \\
=& f(S_t) + f'(S_t)S_t'\Delta + \bar{o}(\Delta)
\end{aligned}
$$

as $\tilde{\Delta}$ and Δ have the same order.

2.3.1 Ito formula for functions of Brownian motion

Now what if we have a function of a Brownian motion? Same thing:

Lemma 2.3.3: Ito formula for functions of Brownian motion

Let $f \in C^2$, then

$$\mathrm{d}f(W_t) = f'(W_t)\,\mathrm{d}W_t + \frac{1}{2}f''(W_t)\,\mathrm{d}t$$

Proof for Lemma

Let $\tilde{\Delta} := W_{t+\Delta} - W_t$, then by finite quadratic variation, $\tilde{\Delta} \overset{n\to\infty}{\to} \mathrm{d}t$

$$
\begin{aligned}
f(W_{t+\Delta}) &= f(W_t + \overbrace{(W_{t+\Delta} - W_t)}^{\tilde{\Delta}}) \\
&= f(W_t) + f'(W_t)\tilde{\Delta} + \frac{1}{2}f''(W_t)\underbrace{\tilde{\Delta}^2}_{\to \mathrm{d}t} + \bar{o}(\tilde{\Delta}^2)
\end{aligned}
$$

Lemma 2.3.4: Ito integral for functions of Brownian motion

In integral notation

$$f(W_T) - f(W_0) = \int_0^t f'(W_s)\,\mathrm{d}W_s + \frac{1}{2}\int_0^t f''(W_s)\,\mathrm{d}s$$

Proof for Lemma

$$
\begin{aligned}
f(W_T) - f(W_0) &= \sum_{i=0}^{n-1} f(W_{t_i^n}) - f(W_{t_i}) \\
&= \sum_{i=0}^{n-1} f'(W_{t_i^n})(W_{t_i^n} - W_{t_i}) + \frac{1}{2}\sum_{i=0}^{n-1} f''(W_{t_i^n})\underbrace{(W_{t_i^n} - W_{t_i})^2}_{\to \mathrm{d}t} + \sum_{i=0}^{n-1} \bar{o}((W_{t_i^n} - W_{t_i})^2) \\
&= \int_0^t f'(W_s)\,\mathrm{d}W_s + \frac{1}{2}\int_0^t f''(W_s)\,\mathrm{d}s + \sum_{i=1}^{n-1} \bar{o}\left((W_{t_i} - W_{t_{i+1}})^2\right)
\end{aligned}
$$

Looking closer at the last term, we can rewrite as:

$$
\sum_{i=1}^{n-1} \bar{o}\left((W_{t_i} - W_{t_{i+1}})^2\right) = \sum_{i=1}^{n-1} \underbrace{\frac{\bar{o}\left((W_{t_i} - W_{t_{i+1}})^2\right)}{(W_{t_i} - W_{t_{i+1}})^2}}_{\to 0} \underbrace{(W_{t_i} - W_{t_{i+1}})^2}_{\to \mathrm{d}t} \overset{n\to\infty}{\to} 0
$$

Thus we are left with the result. ∎

Example.

Let $f(x) = x^2$, then $f'(x) = 2x, f''(x) = 2$ thus

$$
\mathrm{d}f(W_t) = 2W_t\,\mathrm{d}W_t + \mathrm{d}t \implies \int_0^s W_t\,\mathrm{d}W_t = \frac{1}{2}(W_s^2 - s)
$$

(what we found earlier by approximating sequences)

Example.

Let $f(x) = e^{\sigma}x$, then $f'(x) = \sigma e^{\sigma x}, f''(x) = \sigma^2 e^{\sigma x}$ thus

$$
\begin{aligned}
\mathrm{d}e^{\sigma W_t} &= \sigma e^{\sigma W_t}\,\mathrm{d}W_t + \frac{\sigma^2}{2}e^{\sigma W_t}\,\mathrm{d}t \\
\implies e^{\sigma W_t} &= 1 + \int_0^t \sigma e^{\sigma W_s}\,\mathrm{d}W_s + \frac{\sigma^2}{2}\int_0^t e^{\sigma W_s}\,\mathrm{d}s
\end{aligned}
$$

Now suppose f depends on t and W_t, how can we represent it as an integral?

Lemma 2.3.5

Itô formula for $f \in C^{1,2}$ is

$$
\mathrm{d}f(t, W_t) = \left(f_t(t, W_t) + \frac{1}{2}f_{xx}(t, W_t)\right)\mathrm{d}t + f_x(t, W_t)\,\mathrm{d}W_t
$$

(Mnemonic rule is $\mathrm{d}W \cdot \mathrm{d}t = 0, (\mathrm{d}W)^2 = \mathrm{d}t$)

Proof for Lemma

We have

$$
f(t + \Delta, W_{t+\Delta})
$$

> $$
> \begin{aligned}
> &= f(t, W_{t+\Delta}) + f_t(t, W_{t+\Delta})\Delta + \bar{o}(\Delta) \\
> &= f(t, W_t) + f_x(t, W_t)(W_{t+\Delta} - W_t) + \frac{1}{2} f_{xx}(t, W_t)\, \overbrace{(W_{t+\Delta} - W_t)^2}^{\approx\Delta} + \bar{o}(\overbrace{(W_{t+\Delta} - W_t)^2}^{\approx\Delta}) \\
> &\quad + f_t(t, W_t)\Delta + f_{tx}(t, W_t)\, \underbrace{(W_{t+\Delta} - W_t)\Delta}_{\approx\Delta^{3/2}\to 0} + \bar{o}(\Delta) \\
> &= f(t, W_t) + f_x(t, W_t)(W_{t+\Delta} - W_t) + \frac{1}{2} f_{xx}(t, W_t)\Delta + f_t(t, W_t)\Delta + \bar{o}(\Delta)
> \end{aligned}
> $$
>
> Rearrange for $f(t + \Delta, W_{t+\Delta}) - f(t, W_t)$; the limit as $\Delta \to \mathrm{d}t$ yields result. ■

2.3.2 Ito's Lemma in one dimension

We can generalise what we did above for Brownian motion to Itô processes.

Theorem 2.3.6: Itô's Lemma

Suppose $\mathrm{d}X_t = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t$, then

$$\mathrm{d}f(t, X_t) = f_t(t, X_t) + f_x(t, X_t)\,\mathrm{d}X_t + \frac{1}{2} f_{xx}(t, X_t)\,\mathrm{d}\langle X\rangle_t$$

or altenatively

$$\mathrm{d}f(t, X_t) = f_t(t, X_t) + f_x(t, X_t)\,\mathrm{d}X_t + \frac{1}{2} f_{xx}(t, X_t)\sigma_t^2\,\mathrm{d}t$$

Proof. As before with just Brownian motion,

$$
\begin{aligned}
&f(t + \Delta, X_{t+\Delta}) \\
&= f(t, X_{t+\Delta}) + f_t(t, X_{t+\Delta})\Delta + \bar{o}(\Delta) \\
&= f(t, X_t) + f_x(t, X_t)(X_{t+\Delta} - X_t) + \frac{1}{2} f_{xx}(t, X_t)\, \overbrace{(X_{t+\Delta} - X_t)^2}^{\approx\sigma_t^2\Delta} + \bar{o}(\overbrace{(X_{t+\Delta} - X_t)^2}^{\approx\sigma_t^2\Delta}) \\
&\quad + f_t(t, X_t)\Delta + f_{tx}(t, X_t)\, \underbrace{(X_{t+\Delta} - X_t)\Delta}_{\approx\sigma_t\Delta^{3/2}\to 0} + \bar{o}(\Delta) \\
&= f(t, X_t) + f_x(t, X_t)(X_{t+\Delta} - X_t) + \frac{\sigma_t^2}{2} f_{xx}(t, X_t)\Delta + f_t(t, X_t)\Delta + \bar{o}(\Delta)
\end{aligned}
$$

where $(\mathrm{d}X_t)^2 = (\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t)^2$ leaves us with $\sigma^2(\mathrm{d}W_t)^2 = \sigma^2\,\mathrm{d}t$, used in the second line. Rearrange for $f(t + \Delta, X_{t+\Delta}) - f(t, X_t)$; the limit as $\Delta \to \mathrm{d}t$ yields result. □

Example.

> Consider $e^{\int_0^t \sigma_s\,\mathrm{d}W_s}$. Is this a function of W_t? No, instead let $X_t = \int_0^t \sigma_s\,\mathrm{d}W_s$. X_t is an Itô process as
>
> $$X_t = \int_0^t \sigma_s\,\mathrm{d}W_s \iff \mathrm{d}X_t = \sigma_t\,\mathrm{d}W_t$$
>
> Then we have
>
> $$\mathrm{d}e^{X_t} = e^{X_t}\,\mathrm{d}X_t + \frac{\sigma_t^2}{2} e^{X_t}\,\mathrm{d}t$$

2.3.3 Stochastic exponential

Definition 2.3.7: Stochastic exponential

A stochastic exponential is of the form

$$S_t = S_0 \exp\left\{\int_0^t \mu_s \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}W_s\right\}$$

Note that with the substitution $Z_t = \log S_t$ s.t. $\mathrm{d}Z_t = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t$ we can show S_t is an **Itô process**.

Itô's lemma on $S_t = S_0 e^{Z_t}$ implies

$$\mathrm{d}S_t = \alpha_t S_t \,\mathrm{d}t + \sigma_t S_t \,\mathrm{d}W_t$$

where $\alpha_t = \mu_t + \frac{\sigma^2}{2}$, thus S_t is an Itô process. Some additional facts on stochastic exponentials:

- A stochastic exponential is a generalized version of a geometric Brownian motion (i.e. GBM with drift). It is a martingale if $\mu_t = -\frac{\sigma_t^2}{2}$. It can be shown every positive continuous process can be written as a stochastic exponential.
- Stochastic exponentials are used as the stochastic discount factor.

Example.

Now we consider what is the integral representation of $1/S_t$? Note that since $f(x) = 1/x$ is not twice continuously differentiable, Itô formula does not apply in general, however since S_t is strictly positive, and $1/x$ is twice continuously differentiable on $\mathbb{R}_{++}$ everything is ok.

We have $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, thus

$$\begin{aligned}\mathrm{d}\left(\frac{1}{S_t}\right) &= -\frac{1}{S_t^2}\,\mathrm{d}S_t + \frac{1}{2}\frac{2}{S_t^3}(\mathrm{d}S_t)^2 \\ &= -\alpha_t \frac{1}{S_t}\,\mathrm{d}t - \sigma_t \frac{1}{S_t}\,\mathrm{d}W_t + \frac{\sigma_t^2}{S_t}\,\mathrm{d}t\end{aligned}$$

Thus $Y_t := f(S_t) = \frac{1}{S_t}$ is also a stochastic exponential as

$$\mathrm{d}Y_t = (\sigma_t^2 - \alpha_t) Y_t \,\mathrm{d}t - \sigma_t Y_t \,\mathrm{d}W_t$$

2.3.4 Integration by parts (product rule)

Corollary 2.3.8: Itô product rule

Let $\mathrm{d}X_t = \mu_t^X \,\mathrm{d}t + \sigma_t^X \,\mathrm{d}W_t$ and $\mathrm{d}Y_t = \mu_t^Y \,\mathrm{d}t + \sigma_t^Y \,\mathrm{d}W_t$. Then we have

$$\mathrm{d}X_t Y_t = X_t \,\mathrm{d}Y_t + Y_t \,\mathrm{d}X_t + \sigma_t^X \sigma_t^Y \,\mathrm{d}t$$

Proof for Corollary.

Let $\Delta_Z := (Z_{t+\Delta} - Z_t)$ for $Z \in \{X, Y\}$. As before we have

$$\begin{aligned}X_{t+\Delta} Y_{t+\Delta} &= (X_t + \Delta_X)(Y_t + \Delta_Y) + \bar{o}((X_{t+\Delta} - X_t)^2) \\ &= X_t Y_t + X_t \Delta_Y + \Delta_X Y_t + \Delta_X \Delta_Y\end{aligned}$$

Now note that $\Delta_X \approx \mu_t^X \Delta + \sigma_t^X (W_{t+\Delta} - W_t)$ and same for Y, thus

$$\begin{aligned} X_{t+\Delta} Y_{t+\Delta} = {} & X_t Y_t + X_t \Delta_Y + \Delta_X Y_t \\ & + (\mu_t^X \Delta + \sigma_t^X (W_{t+\Delta} - W_t))(\mu_t^Y \Delta + \sigma_t^Y (W_{t+\Delta} - W_t)) \end{aligned}$$

Removing higher order terms (i.e. $\Delta^p; p > 1$), we have:

$$X_{t+\Delta} Y_{t+\Delta} = X_t Y_t + X_t \Delta_X + Y_t \Delta_Y + \sigma_t^X \sigma_t^Y \Delta$$

Rearrange for $X_{t+\Delta} Y_{t+\Delta} - X_t Y_t$; the limit as $\Delta \to \mathrm{d}t$ yields result. ■

2.4 Exercises

2.5 Solutions

Chapter 3

Valuation

3.1 Financial Markets

3.1.1 Primary Assets

Definition 3.1.1: Bond price model

Assume money-market follows:

$$\mathrm{d}B_t = B_t r \,\mathrm{d}t; \quad B_0 = 1 \tag{3.1.1}$$

where r is a constant interest rate. The solution is given by

$$B_t = e^{rt}$$

Definition 3.1.2: Stock price model

Assume stock price follows a GBM

$$\mathrm{d}S_t = S_t \mu \,\mathrm{d}t + S_t \sigma \,\mathrm{d}W_t; S_0 > 0 \tag{3.1.2}$$

The solution is given by

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t}$$

3.1.2 Derivatives

Definition 3.1.3: American Options

Definition 3.1.4: European Options

Definition 3.1.5: Down-and-Out European Options

Definition 3.1.6: Arithmetic-Average Asian Options

3.1.3 Self-Financing Portfolios

Definition 3.1.7: Portfolio Wealth

Let N_t^0 denote value in the money market, and N_t shares of stock. The value of our portfolio at time t is given by

$$X_t = N_t^0 B_t + N_t S_t$$

In addition, assume we start with initial capital X_0.

At time 0, we have

$$X_0 = N_0^0 \underbrace{B_0}_{=1} + N_0 S_0 = N_0^0 + N_0 S_0$$

In discrete time, assume the investor changes portfolio positions at discrete times only, with no transaction costs. Let the next re-balancing occur at time $s > t$. Just before the rebalancing at time $s-$, our portfolio has value:

$$\begin{aligned} X_{s-} &= N_t^0 B_{s-} + N_t S_{s-} \\ &= N_t^0 B_s + N_t S_s \end{aligned}$$

where the second line comes from the continuity of B_t, S_t. The portfolio has wealth X_{s+} just after rebalancing, defined as:

$$X_{s+} = N_s^0 B_s + N_s S_s$$

This lends itself to the next definition:

Definition 3.1.8: Self-financing portfolio

Let X_{t-}, X_{t+} represent the portfolio's value prior to and just after the rearrangement of positions at time t. Self-financing means that

$$X_{t-} = X_{t+}$$

i.e. rebalancing does not change investors wealth.

If X_t is self-financing, then we can write:

$$\begin{aligned} N_t^0 B_s + N_t S_s &= N_s^0 B_s + N_s S_s \\ \iff (N_t^0 - N_s^0) B_s + (N_t - N_s) S_s &= 0 \end{aligned}$$

Taking the intervals between s and t smaller, then we have:

$$B_t \, \mathrm{d}N_t^0 + S_t \, \mathrm{d}N_t = 0$$

Lemma 3.1.9: Self-financing portfolio dynamics in continuous time

It holds that

$$\mathrm{d}X_t = N_t^0 \,\mathrm{d}B_t + N_t \,\mathrm{d}S_t$$

Proof for Lemma

> It may seem trivial, but we cannot simply differentiate both sides of $X_t = N_t^0 B_t + N_t S_t$. Instead, again consider the time now, t, and some time ahead $s > t$. The change in portfolio wealth is
>
> $$X_{s-} - X_t = N_t^0(B_s - B_t) + N_t(S_s - S_t)$$
>
> Since X_t is self-financing, we have $X_s = X_{s-}$ therefore
>
> $$X_s - X_t = N_t^0(B_s - B_t) + N_t(S_s - S_t)$$
>
> Therefore, changes in portfolio wealth in between rebalancing can only be due to changes in the asset prices. When we take $s - t \to 0$, then
>
> $$\mathrm{d}X_t = N_t^0 \,\mathrm{d}B_t + N_t \,\mathrm{d}S_t$$
>
> ■

If we substitute the processes (3.1.1) and (3.1.2) into $\mathrm{d}X_t$, we get:

$$\begin{aligned} \mathrm{d}X_t &= N_t^0 r \,\mathrm{d}t + N_t(\mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t) \\ &= X_t r \,\mathrm{d}t + (\mu - r) N_t S_t \,\mathrm{d}t + \sigma N_t S_t \,\mathrm{d}W_t \end{aligned} \tag{3.1.3}$$

Definition 3.1.10: Portfolio Process

More often we work with the value of our positions, π_t which we call the portfolio process. It is defined as:

$$\pi_t = N_t S_t$$

We can rewrite (3.1.3) as

$$\mathrm{d}X_t = X_t r \,\mathrm{d}t + \pi_t(\mu - r) \,\mathrm{d}t + \sigma \pi_t \,\mathrm{d}W_t$$

The solution can be shown as

$$X_t = e^{rt}\left(X_0 + (\mu - r)\int_0^t e^{-ru}\pi_u \,\mathrm{d}u + \sigma \int_0^t e^{-ru}\pi_u \,\mathrm{d}W_u\right) \tag{3.1.4}$$

3.2 The idea behind valuation

3.2.1 Replication and no-arbitrage

If we can construct a portfolio that has exactly the same payoff at time T, then they must have the same price for all $t \leq T$. If this is not true, then one can earn riskless profits from this *arbitrage* opportunity.

Definition 3.2.1: Arbitrage

Given a trading strategy π_t, and portfolio wealth $X_t(X_0, \pi_t)$, we call it an arbitrage if

- $X_0 < 0$ and for all $\omega \in \Omega, X_T(\omega) \geq 0$
- or $X_0 = 0$ and
  - for all $\omega \in \Omega, X_T(\omega) \geq 0$
  - for at least one $\omega \in \Omega$ $X_T(\omega) > 0$

We now consider a famous no-arbitrage condition that relates the option prices to the stock price.

Theorem 3.2.2: Put-Call Parity

Consider a European call and put option with price C_t, P_t, both with the same maturity T, strike price K. Then the following relation holds:

$$C_t - P_t = S_t - Ke^{-r(T-t)} \tag{3.2.1}$$

Proof. Consider a portfolio at time T,

$$\begin{aligned}
\pi_T &= K - S_T + C_T - P_T \\
&= K - S_T + (S_T - K)^+ - (K - S_T)^+ \\
&= K - S_T + \begin{Bmatrix} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T < K \end{Bmatrix} - \begin{cases} 0 & \text{if } S_T > K \\ K - S_T & \text{if } S_T < K \end{cases} \\
&= K - S_T + \begin{cases} S_T - K & \text{if } S_T > K \\ S_T - K & \text{if } S_T < K \end{cases} \\
&= 0
\end{aligned}$$

Since they have a guaranteed payoff of zero at T, the price must be the same, i.e.

$$\begin{aligned}
\pi_t &= Ke^{-r(T-t)} - S_t + C_t - P_t = 0 \\
\iff C_t - P_t &= S_t - Ke^{-r(T-t)}
\end{aligned}$$

□

On **replication more generally**, suppose we know that there exists a self-financing portfolio π s.t. final wealth at T is equal to the payoff of asset H at T

$$X_T(X_0, \pi_T) = H_T \quad \mathbb{P}\text{-a.s.}$$

Then by no arbitrage, the price of both assets must be the same at all times prior. In particular, due to (3.1.4)

$$H_T = e^{rT} \left(X_0 + (\mu - r) \int_0^T e^{-ru} \pi_u \, \mathrm{d}u + \sigma \int_0^T e^{-ru} \pi_u \, \mathrm{d}W_u \right)$$

If we are to recover X_0, then we need to find a π that replicates the payoff.

$$\begin{aligned} e^{-rT} H_T &= X_0 + \int_0^T e^{-ru} \pi_u \sigma \frac{\mu - r}{\sigma} \, \mathrm{d}u + \int_0^T e^{-ru} \pi_u \sigma \, \mathrm{d}W_u \\ &= X_0 + \int_0^T e^{-ru} \pi_u \sigma \underbrace{\left(\frac{\mu - r}{\sigma} \, \mathrm{d}u + \mathrm{d}W_u \right)}_{= \mathrm{d}W_u^{\vartheta}} \\ &= X_0 + \sigma \int_0^T e^{-ru} \pi_u \, \mathrm{d}W_u^{\vartheta} \end{aligned}$$

We need an equivalent measure s.t. $H_T = X_T(X_0, \pi) - \mathbb{P}$ a.s., so the part on the right can be written as a Brownian motion under this measure. Assume there exists a measure $\mathbb{Q}$ s.t.

$$\mathrm{d}W_t^{\vartheta} = \frac{\mu - r}{\sigma} \, \mathrm{d}t + \mathrm{d}W_t$$

is a Brownian motion. Under sufficient conditions for π_t (that is $\pi_t \in \mathcal{H}^2$) then the Itô integral w.r.t. a Brownian motion is a martingale, $\mathbb{E}^{\mathbb{Q}} \left[I_T | \mathcal{F}_0 \right] = I_0 = 0$. By taking conditional expectations,

$$X_0 = e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[H_T | \mathcal{F}_0 \right]$$

3.2.2 Risk-neutral valuation

Definition 3.2.3: Market price of risk

Define the *market price of risk*, ϑ as

$$\vartheta := \frac{\mu - r}{\sigma}$$

Consider the exponential martingale L given by

$$\mathrm{d}L_t = \vartheta L_t \, \mathrm{d}W_t; \quad L_0 = 1$$

By Itô product rule,

$$L_t = \exp \left(-\frac{1}{2} \vartheta^2 t - \vartheta W_t \right)$$

Definition 3.2.4: Equivalent Martingale Measure

We define the *risk-neutral* or *equivalent martingale* probability measure, $\mathbb{Q}_T$ on the measurable space $(\Omega, \mathcal{F}_T)$ as: for $A \in \mathcal{F}_T$,

$$\mathbb{Q}_T(A) = \mathbb{E}^{\mathbb{P}} \left[L_T \mathbf{1}_{\{A\}} \right]$$

We often will refer to this as just $\mathbb{Q}$.

Theorem 3.2.5: Girsanov's Theorem

Under the measure $\mathbb{Q}$, the process $(W_t^\vartheta; t \in [0,T])$ given by

$$W_t^\vartheta = \vartheta t + W_t$$

is a $(\mathcal{F}_t)$-Brownian motion.

Proof. We show that under measure $\mathbb{Q}$, $W_t^\vartheta := \vartheta t + W_t$ is a Brownian motion. First note that W_t^ϑ is continuous as W_t is continuous, and $W_0^\vartheta = 0$. Now we need to show that increments are **normally distributed** and **independent** under $\mathbb{Q}$.

Partition $[0,T]$ into

$$0 = t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n = T$$

Consider the joint probability function under $\mathbb{Q}_T$

$$\begin{aligned}&\mathbb{Q}_T(W_{t_1}^\vartheta - W_{t_0}^\vartheta \leq x_1, \ldots, W_{t_n}^\vartheta - W_{t_{n-1}}^\vartheta \leq x_n)\\ =\ &\mathbb{Q}_T(W_{t_1} - W_{t_0} \leq x_1 - \vartheta(t_1 - t_0), \ldots, W_{t_n} - W_{t_{n-1}} \leq x_n - \vartheta(t_n - t_{n-1}))\end{aligned}$$

which follows by substituting the definition of W_t^ϑ above. By the definition of the measure $\mathbb{Q}$, the above is equal to

$$\begin{aligned}&= \mathbb{E}^{\mathbb{P}}\left[L_T \mathbf{1}_{\{W_{t_1} - W_{t_0} \leq x_1 - \vartheta(t_1 - t_0), \ldots, W_{t_n} - W_{t_{n-1}} \leq x_n - \vartheta(t_n - t_{n-1})\}}\right]\\ &= \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\frac{1}{2}\vartheta^2 T - \vartheta W_T\right)\prod_{i=1}^n \mathbf{1}_{\{W_{t_i} - W_{t_{i-1}} \leq x_i - \vartheta(t_i - t_{i-1})\}}\right]\\ &= \exp\left(-\frac{1}{2}\vartheta^2 T\right)\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\vartheta W_T\right)\prod_{i=1}^n \mathbf{1}_{\{W_{t_i} - W_{t_{i-1}} \leq x_i - \vartheta(t_i - t_{i-1})\}}\right]\end{aligned}$$

Note that we can rewrite $W_T = \sum_{i=1}^n W_{t_i} - W_{t_{i-1}}$ as a telescopic sum. Thus we have

$$\begin{aligned}&= \exp\left(-\frac{1}{2}\vartheta^2 T\right)\mathbb{E}^{\mathbb{P}}\left[\prod_{i=1}^n \exp\left(-\vartheta(W_{t_i} - W_{t_{i-1}})\right)\mathbf{1}_{\{W_{t_i} - W_{t_{i-1}} \leq x_i - \vartheta(t_i - t_{i-1})\}}\right]\\ &= \exp\left(-\frac{1}{2}\vartheta^2 T\right)\prod_{i=1}^n \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\vartheta(W_{t_i} - W_{t_{i-1}})\right)\mathbf{1}_{\{W_{t_i} - W_{t_{i-1}} \leq x_i - \vartheta(t_i - t_{i-1})\}}\right]\end{aligned}$$

as $W_{t_i} - W_{t_{i-1}}$ are IID normal (independent). Now we focus on the i-th term in the product; using the normal MGF, $\mathbb{E}^{\mathbb{P}}\left[e^{W_{t_i} - W_{t_{i-1}}}\right]$ we have

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\vartheta(W_{t_i} - W_{t_{i-1}})\right)\mathbf{1}_{\{W_{t_i} - W_{t_{i-1}} \leq x_i - \vartheta(t_i - t_{i-1})\}}\right]\\ =\ &\int_{-\infty}^{x_i - \vartheta(t_i - t_{i-1})} \exp(-\vartheta x)\frac{1}{\sqrt{t_i - t_{i-1}}\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{x^2}{t_i - t_{i-1}}\right)\mathrm{d}x\end{aligned}$$

which when rearranged gives

$$= \frac{1}{\sqrt{t_i - t_{i-1}}\sqrt{2\pi}}\int_{-\infty}^{x_i - \vartheta(t_i - t_{i-1})} \exp\left(-\vartheta x - \frac{1}{2}\frac{x^2}{t_i - t_{i-1}}\right)\mathrm{d}x$$

Note the exponential can be rewritten as

$$\begin{aligned}
&\exp\left(-\vartheta x - \frac{1}{2}\frac{x^2}{t_i - t_{i-1}}\right)\\
&= \exp\left(-\frac{1}{2}\frac{2\vartheta x(t_i - t_{i-1}) + x^2}{t_i - t_{i-1}}\right)\\
&= \exp\left(-\frac{1}{2}\frac{x^2 + 2\vartheta x(t_i - t_{i-1}) + [\vartheta^2(t_i - t_{i-1})^2 - \vartheta^2(t_i - t_{i-1})^2]}{t_i - t_{i-1}}\right)\\
&= \exp\left(-\frac{1}{2}\frac{(x + \vartheta(t_i - t_{i-1}))^2}{t_i - t_{i-1}} + \frac{1}{2}\vartheta^2(t_i - t_{i-1})\right)
\end{aligned}$$

Thus putting it back into the integral, we get

$$\frac{\exp(\frac{1}{2}\vartheta^2(t_i - t_{i-1}))}{\sqrt{t_i - t_{i-1}}\sqrt{2\pi}}\int_{-\infty}^{x_i - \vartheta(t_i - t_{i-1})} \exp\left(-\frac{1}{2}\frac{(x + \vartheta(t_i - t_{i-1}))^2}{t_i - t_{i-1}}\right)\mathrm{d}x$$

Using a change of variable $y := x + \vartheta(t_i - t_{i-1})$, we have

$$\frac{\exp(\frac{1}{2}\vartheta^2(t_i - t_{i-1}))}{\sqrt{t_i - t_{i-1}}\sqrt{2\pi}}\int_{-\infty}^{x_i} \exp\left(-\frac{1}{2}\frac{y^2}{t_i - t_{i-1}}\right)\mathrm{d}y$$

Now plug this back into the the product

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\vartheta^2 T\right)\prod_{i=1}^{n}\exp\left(\frac{1}{2}\vartheta^2(t_i - t_{i-1})\right)\int_{-\infty}^{x_i}\frac{\exp\left(-\frac{1}{2}\frac{y^2}{t_i - t_{i-1}}\right)}{\sqrt{t_i - t_{i-1}}\sqrt{2\pi}}\mathrm{d}y\\
&= \exp\left(-\frac{1}{2}\vartheta^2 T\right)\prod_{i=1}^{n}\exp\left(\frac{1}{2}\vartheta^2(t_i - t_{i-1})\right)\prod_{i=1}^{n}\int_{-\infty}^{x_i}\frac{\exp\left(-\frac{1}{2}\frac{y^2}{t_i - t_{i-1}}\right)}{\sqrt{t_i - t_{i-1}}\sqrt{2\pi}}\mathrm{d}y\\
&= \exp\left(-\frac{1}{2}\vartheta^2 T\right)\exp\left(\frac{1}{2}\vartheta^2 T\right)\prod_{i=1}^{n}\int_{-\infty}^{x_i}\frac{\exp\left(-\frac{1}{2}\frac{y^2}{t_i - t_{i-1}}\right)}{\sqrt{t_i - t_{i-1}}\sqrt{2\pi}}\mathrm{d}y\\
&= \prod_{i=1}^{n}\int_{-\infty}^{x_i}\frac{1}{\sqrt{t_i - t_{i-1}}\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{y^2}{t_i - t_{i-1}}\right)\mathrm{d}y
\end{aligned}$$

which is IID normal with mean zero and variance $t_i - t_{i-1}$ □

Corollary 3.2.6

For any $Z \in \mathcal{F}_T$ and $s \leq T$ we have

$$\mathbb{E}^{\mathbb{Q}}[Z|F_s] = \frac{\mathbb{E}^{\mathbb{P}}[L_T Z|\mathcal{F}_s]}{L_s}$$

Proof for Corollary.

> By definition of conditional expectation, we want to show that for any $A \in \mathcal{F}_s$,
>
> $$\mathbb{E}^{\mathbb{Q}}\left[Z\mathbf{1}_{\{A\}}\right] = \mathbb{E}^{\mathbb{Q}}\left[Y\mathbf{1}_{\{A\}}\right]$$
>
> where Y is the RHS. By definition of the $\mathbb{Q}$-measure
>
> $$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[Z\mathbf{1}_{\{A\}}\right] &= \mathbb{E}^{\mathbb{P}}\left[L_T Z\mathbf{1}_{\{A\}}\right]\\ &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[L_T Z\mathbf{1}_{\{A\}}|\mathcal{F}_s\right]\right]\end{aligned}$$

$$= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[L_T Z|\mathcal{F}_s\right]\mathbf{1}_{\{A\}}\right]$$
$$= \mathbb{E}^{\mathbb{P}}\left[L_s \frac{\mathbb{E}^{\mathbb{P}}\left[L_T Z|\mathcal{F}_s\right]\mathbf{1}_{\{A\}}}{L_s}\right]$$
$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{A\}} \frac{\mathbb{E}^{\mathbb{P}}\left[L_T Z|\mathcal{F}_s\right]}{L_s}\right]$$

Remark.

> Let $Y \in \mathcal{F}_s$ then $\mathbb{E}^{\mathbb{Q}}[Y] = \mathbb{E}^{\mathbb{P}}[L_T Y] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}[L_T Y|\mathcal{F}_s]] = \mathbb{E}^{\mathbb{P}}\left[Y\mathbb{E}^{\mathbb{P}}[L_T|\mathcal{F}_s]\right] = \mathbb{E}^{\mathbb{P}}[Y L_s]$, i.e. if a random variable is measurable w.r.t. $\mathcal{F}_s$, then we can change the measure with $\mathcal{L}_s$

3.3 Formal valuation in a Black-Scholes market

Definition 3.3.1: Black-Scholes market

In a *Black-Scholes market*, we assume

1. the stock price follows a GBM
2. no arbitrage opportunities
3. market completeness

Theorem 3.3.2: Black-Scholes martingale processes

In the Black-Scholes model, under the EMM $\mathbb{Q}$

(i) the discounted stock price process, $\{e^{-rt}S_t\}_{t\in[0,T]}$ is a $(\mathcal{F}_t)$-martingale

(ii) the discounted portfolio wealth process $\{e^{-rt}X_t\}_{t\in[0,T]}$ is a $(\mathcal{F}_t)$-local martingale

Proof. (i) Using the integration by parts formula on $f(t, S_t) = e^{-rt}S_t$ we have

$$\begin{aligned}
\mathrm{d}(e^{-rt}S_t) &= e^{-rt}\,\mathrm{d}S_t + S_t\,\mathrm{d}(e^{-rt}) \\
&= e^{-rt}(\mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t) - rS_t e^{-rt}\,\mathrm{d}t \\
&= e^{-rt}S_t\sigma\left(\frac{\mu - r}{\sigma}\,\mathrm{d}t + \mathrm{d}W_t\right) \\
&= e^{-rt}S_t\sigma\left(\vartheta\,\mathrm{d}t + \mathrm{d}W_t\right) \\
&= e^{-rt}S_t\sigma\,\mathrm{d}W_t^{\vartheta}
\end{aligned}$$

The solution (using the stochastic exponential solution with $\mu = 0$) is

$$e^{-rt}S_t = S_0\exp\left(-\frac{1}{2}\sigma^2 t + \sigma\,\mathrm{d}W_t^{\vartheta}\right)$$

Recall from the last exercise in chapter 1, that the RHS is a $(\mathcal{F}_t)$-martingale; equating LHS and RHS follows the result.

(ii) Similarly for a self-financing portfolio, we have for $t \in [0, T]$

$$e^{-rt}X_t = X_0 + (\mu - r)\int_0^t e^{-ru}\pi_u\,\mathrm{d}u + \sigma\int_0^t e^{-ru}\pi_u\,\mathrm{d}W_u$$

$$
\begin{aligned}
&= X_0 + \sigma\vartheta \int_0^t e^{-ru}\pi_u \,\mathrm{d}u + \sigma \int_0^t e^{-ru}\pi_u \,\mathrm{d}W_u \\
&= X_0 + \sigma \int_0^t e^{-ru}\pi_u(\vartheta \,\mathrm{d}u + \mathrm{d}W_u) \\
&= X_0 + \sigma \int_0^t e^{-ru}\pi_u \,\mathrm{d}W_u^{\vartheta}
\end{aligned}
$$

In general, the Itô integral w.r.t. a Brownian motion is a local martingale (to show it is a martingale requires additional restrictions on π_t).

□

3.3.1 Admissibility and Arbitrage

Definition 3.3.3: Admissible

A self-financing portfolio strategy is *admissible* if its associated wealth process X is bounded from below by a constant, which may depend on the portfolio itself.

Theorem 3.3.4: Admissible $\implies$ no-arbitrage

In the Black-Scholes model, there is no admissible self-financing portfolio that offers arbitrage opportunities.

Proof. Assume $\exists \pi_t$ which is admissible, with wealth process X

- $X_0 = 0$
- arbitrage:
  - for some $T > 0$ $\mathbb{P}\left(X_T \geq 0\right) = 1$
  - for some event $A \in \mathcal{F}_t$ with $\mathbb{P}\left(A\right) > 0, X_T(\omega) > 0$ for all $\omega \in A$.

Part 1. Consider the risk-neutral measure $\mathbb{Q}$ which is equivalent a.s. to $\mathbb{P}$, that is

$$
\begin{aligned}
\mathbb{P}\left(C\right) = 0 &\iff \mathbb{Q}(C) = 0 \\
\mathbb{P}\left(C\right) = 1 &\iff \mathbb{Q}(C) = 1
\end{aligned}
$$

Equivalence implies the arbitrage conditions becomes:

- $\mathbb{Q}\left(X_T \geq 0\right) = 1$
- for some $A \in \mathcal{F}_T$ with $\mathbb{Q}\left(A\right) > 0$, $X_T(\omega) > 0$ for all $\omega \in A$.

Therefore it follows from this that $\mathbb{E}^{\mathbb{Q}}\left[X_T(\omega)\right] > 0$.

Part 2. Under the risk-neutral measure, the discounted portfolio value process follows

$$
e^{-rt}X_t = \sigma \int_0^t e^{-rs}\pi_s \,\mathrm{d}W_s^{\vartheta}
$$

Since we assume the portfolio strategy is admissible, therefore there exists a constant $K \geq 0$ s.t. $e^{-rt}X_t \geq K$ for all $t \in [0, T]$ (bounded below). Thus $e^{-rt}X_t + K \geq 0$ is a positive $(\mathcal{F}_t)$-local martingale w.r.t. the measure $\mathbb{Q}$.

Positive local martingales are supermartingales (see this chapter exercises for proof), thus $e^{-rt}X_t + K$ is an $(\mathcal{F}_t)$-supermartingale w.r.t. the probability measure $\mathbb{Q}$ i.e. for $t \geq s$

$$\mathbb{E}^{\mathbb{Q}}\left[e^{rt}X_t + K|\mathcal{F}_s\right] \leq e^{rs}X_s + K$$

We have for X_T:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[X_T\right] &= e^{-rT}(e^{-rT}X_T + K - K) \\
&= e^{rT}\left(\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}X_T + K\right] - K\right) \\
&= e^{rT}\left(\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}X_T + K|\mathcal{F}_0\right]\right] - K\right) \\
&\leq e^{rT}\left(\mathbb{E}^{\mathbb{Q}}\left[X_0 + K\right] - K\right) \\
&= 0
\end{aligned}$$

However note the results in Parts 1 and 2 derived independently from each other contradict, therefore there cannot be an arbitrage. □

3.3.2 Formal Valuation Theorems

Definition 3.3.5: European Contingent Claim

Definition 3.3.6: Replicating Process

Theorem 3.3.7: Existence and pricing of European claims

Consider a European contingent claim with payoff $H_T \in \mathcal{L}^1(\Omega, \mathcal{F}_T, \mathbb{Q})$ bounded from below, where $\mathbb{Q}$ is the EMM. Then there exists a replicating portfolio s.t. $X_T = H_T$. Moreover, for $t \in [0, T]$ the no-arbitrage price of H_T is

$$P_t^{H_T} = e^{-r(T-t)}\mathbb{E}^{\mathbb{P}}\left[\frac{L_T}{L_t}H_T\middle|\mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[H_T|\mathcal{F}_t\right]$$

Motivation for Proof. Remember that if we can find a replicating portfolio π_t s.t. the associated wealth process

$$X_T(X_0, \pi) = H_T$$

then the value of the portfolio is the same as the asset H_t at time $t \leq T$, that is

$$e^{-rt}X_t^{\pi} = \frac{1}{L_t}\mathbb{E}^{\mathbb{P}}\left[e^{-rT}L_T H_T|\mathcal{F}_t\right]$$

Now consider the $(\mathcal{F}_t, \mathbb{P})$-martingale M defined by

$$M_t := \mathbb{E}^{\mathbb{P}}\left[e^{-rT}L_T H_T|\mathcal{F}_t\right]$$

s.t. $e^{-rt}X_t = \frac{M_t}{L_t}$. Then by the martingale representation theorem, there exists a K that is $(\mathcal{F}_t)$-adapted s.t.

for all $t \geq 0$

$$M_t = \mathbb{E}^{\mathbb{P}}\left[e^{-rT}L_T H_T\right] + \int_0^t K_s \,\mathrm{d}W_s \iff \mathrm{d}M_t = K_t \,\mathrm{d}W_t = K_t(\mathrm{d}W_t^{\vartheta} - \theta \,\mathrm{d}t)$$

Now recall that $\mathrm{d}L_t = -\theta L_t \,\mathrm{d}W_t$, which by Itô's lemma implies

$$\mathrm{d}L_t^{-1} = \theta^2 L_t^{-1} \,\mathrm{d}t + \theta L_t^{-1} \,\mathrm{d}W_t = \theta L_t^{-1} \,\mathrm{d}W_t^{\vartheta}$$

Integrating by parts, we have

$$\begin{aligned}
\mathrm{d}\left(\frac{M_t}{L_t}\right) &= M_t \,\mathrm{d}L_t^{-1} + L_t^{-1} \,\mathrm{d}M_t + \theta K_t L_t^{-1} \,\mathrm{d}t \\
&= M_t \theta L_t^{-1} \,\mathrm{d}W_t^{\vartheta} + L_t^{-1} K_t \,\mathrm{d}W_t^{\vartheta} - L_t^{-1} \theta K_t \,\mathrm{d}t + \theta K_t L_t^{-1} \,\mathrm{d}t \\
&= (M_t \theta + K_t) L_t^{-1} \,\mathrm{d}W_t^{\vartheta}
\end{aligned}$$

and

$$\frac{M_t}{L_t} = \frac{M_0}{L_0} + \int_0^t \frac{M_s \theta + K_s}{L_s} \,\mathrm{d}W_s^{\vartheta}$$

Proof. Consider the self-financing strategy π_t with initial wealth

$$X_0 = M_0 \qquad \text{(which is also just } \frac{M_0}{L_0}\text{)}$$

and process defined by

$$\pi_t := e^{rt}\left(\frac{M_t \theta + K_t}{\sigma L_t}\right)$$

Therefore using the portfolio evolution definition, the discounted wealth process can be written as

$$e^{-rt}X_t = \frac{M_0}{L_0} + \int_0^t \frac{M_s \theta + K_s}{L_s} \,\mathrm{d}W_s^{\vartheta}$$

In the motivation for the proof, we show that the RHS can be expressed simply as M_t/L_t through repeated application of Itô calculus. Therefore at time T, we have that X_T by our definition of M_t is

$$\begin{aligned}
X_T &= e^{rT}\frac{M_T}{L_T} \\
&= \frac{e^{rT}}{L_T}\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}L_T H_T | \mathcal{F}_T\right] \\
&= H_T \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

Therefore since there exists a replicating portfolio (note that turns out to be a true martingale), then the price at time t, H_t must be equal to the wealth process at time t, X_t. ☐

Corollary 3.3.8: Price of European options

In a Black-Scholes market, the no-arbitrage price process of a European contingent claim that yields the payoff $F(S_T) \in \mathcal{L}^1(\Omega, \mathcal{F}_t, \mathbb{Q})$ at maturity $T > 0$ is given by

$$P_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}F(S_T)|\mathcal{F}_t\right]$$

for $t \in [0, T]$, where

$$\frac{\mathrm{d}S_t}{S_t} = r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\vartheta}$$

and W^{ϑ} is an $(\mathcal{F}_t)$-Brownian motion under the risk-neutral (EMM) measure $\mathbb{Q}$.

Example.

> Consider a European vanilla call option with maturity T and strike price K. The no-arbitrage price, C_t^{vanilla} at time t is given by
>
> $$\begin{aligned} C_t^{\text{vanilla}} &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+|\mathcal{F}_t\right] \\ &= S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2) \end{aligned}$$
>
> where
>
> $$d_1 := \frac{\log\left(S_t/K\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 := d_1 - \sigma\sqrt{T-t}$$

Proof. Under the risk-neutral measure, the stock price at time t is given by

$$S_T = S_t \exp\left(r - \frac{1}{2}\sigma^2(T-t) + \sigma\sqrt{T-t}\frac{W_T^{\vartheta} - W_t^{\vartheta}}{\sqrt{T-t}}\right)$$

...tbc... □

Definition 3.3.9: Stochastic Discount Factor (SDF)

We define the *stochastic discount factor* or *state price density*, ξ_t as

$$\xi_t := e^{-rt}L_t \quad \text{where} \quad L_t := \mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|\mathcal{F}_t\right]$$

In complete markets, the no-arbitrage price of payoff H_T can be expressed as

$$P_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}H_T|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{\xi_T}{\xi_t}H_T|\mathcal{F}_t\right]$$

We can therefore interpret L_T as the price of risk for the T-horizon.

3.4 Exercises

3.5 Solutions

Ex.1 In a standard Black-Scholes market, the stock price follows

$$\mathrm{d}S_t = rS_t + \sigma S_t \,\mathrm{d}W_t^{\vartheta}$$

where W_t^{ϑ} is a Brownian motion under the equivalent martingale measure. The solution is given by

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t r - \frac{1}{2}\sigma^2 \,\mathrm{d}u + \int_0^t \sigma \,\mathrm{d}W_u^{\vartheta}\right) \\ &= S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma W_t^{\vartheta}\right) \end{aligned}$$

and thus

$$S_t^p = S_0^p \exp\left(\left(r - \frac{1}{2}\sigma^2\right) pt + p\sigma W_t^{\vartheta}\right)$$

In a Black-Scholes market, the price of the option is given by the formula

$$\begin{aligned} P_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T^p - K)^+ | \mathcal{F}_0\right] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[S_T^p \mathbf{1}_{\{S_T^p > K\}} - K\mathbf{1}_{\{S_T^p > K\}} | \mathcal{F}_0\right] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[S_T^p \mathbf{1}_{\{S_T^p > K\}} | \mathcal{F}_0\right] - e^{-rT}K\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{S_T^p > K\}} | \mathcal{F}_0\right] \end{aligned}$$

The inequality can be rewritten as

$$\begin{aligned} & S_T^p > K \\ \Longleftrightarrow\ & S_0^p \exp\left(\left(r - \frac{1}{2}\sigma^2\right) pT + p\sigma W_T^{\vartheta}\right) > K \\ \Longleftrightarrow\ & \frac{W_T^{\vartheta}}{\sqrt{T}} > \frac{\log \frac{K}{S_0^p} - \left(r - \frac{1}{2}\sigma^2\right) pT}{p\sigma\sqrt{T}} =: -d_2 \end{aligned}$$

where $W_T^{\vartheta}/\sqrt{T} \sim N(0,1)$.

Using the fact $\mathbb{E}^{\mathbb{Q}}\left[e^{a+bZ}\mathbf{1}_{\{Z>c\}}\right] = \exp(a + b^2/2)\Phi(b - c)$ for standard normal Z, we have

$$\begin{aligned} e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[S_T^p \mathbf{1}_{\{S_T^p > K\}} | \mathcal{F}_0\right] &= e^{-rT} S_0^p \exp\left(\left(r - \frac{1}{2}\sigma^2\right) pT + p^2\sigma^2 T\right) \Phi\left(p\sigma\sqrt{T} - d_2\right) \\ &= \exp\left((p-1)rT - \frac{1}{2}\sigma^2 pT\right) S_0^p \Phi(d_1) \end{aligned}$$

where $d_1 := p\sigma\sqrt{T} - d_2$. Similarly we have $\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{Z>c\}}\right] = \Phi(-c)$, thus

$$e^{-rT}K\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{S_T^p > K\}} | \mathcal{F}_0\right] = e^{-rT}K\Phi(d_2)$$

Putting both parts together, we have

$$P_0 = \exp\left((p-1)rT - \frac{1}{2}\sigma^2 pT\right) S_0^p \Phi(d_1) + e^{-rT}K\Phi(d_2)$$

Ex.2 Consider $\ln S_t$. By Itô's lemma,

$$\begin{aligned} \mathrm{d}\ln S_t &= \frac{1}{S_t}\,\mathrm{d}S_t - \frac{1}{2}\frac{1}{S_t^2}(\mathrm{d}S_t)^2 \\ &= \frac{1}{S_t}(\mu_t S_t\,\mathrm{d}t + \sigma_t S_t\,\mathrm{d}W_t) - \frac{1}{2}\frac{1}{S_t^2}\sigma_t^2 S_t^2\,\mathrm{d}t \\ &= \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t - \frac{1}{2}\sigma_t^2\,\mathrm{d}t \\ &= \left(\mu_t - \frac{1}{2}\sigma_t^2\right)\mathrm{d}t + \sigma_t\,\mathrm{d}W_t \end{aligned}$$

The solution to this is

$$\ln S_t = \ln S_0 + \int_0^t \mu_u - \frac{1}{2}\sigma_u^2\,\mathrm{d}u + \int_0^t \sigma_u\,\mathrm{d}W_u$$

and therefore

$$S_t = S_0 \exp\left(\int_0^t \mu_u - \frac{1}{2}\sigma_u^2\,\mathrm{d}u + \int_0^t \sigma_u\,\mathrm{d}W_u\right)$$

Ex.3 By the definition of a local martingale, there exists a sequence of stopping times $\tau_n \uparrow T$ s.t.

$$M_t^n := X_{t\wedge\tau_n} \geq 0$$

is a martingale for any n. For $s \leq t$, the martingale property is

$$\mathbb{E}\left[M_t^n|\mathcal{F}_s\right] = M_s^n$$

Next we apply Fatou's Lemma as $M_t^n \geq 0$ for all $t \in [0,T]$

$$\mathbb{E}\left[\liminf_{n\to\infty} M_t^n|\mathcal{F}_s\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[M_t^n|\mathcal{F}_s\right]$$

Note that as $n \to \infty$, $M_t^n = X_t$ which means the LHS converges to $\mathbb{E}\left[X_t|\mathcal{F}_s\right]$. By the same logic, the RHS is equal to X_s as shown below on the right

$$\begin{aligned} \mathbb{E}\left[X_t|\mathcal{F}_s\right] &\leq \liminf_{n\to\infty} \mathbb{E}\left[M_t^n|\mathcal{F}_s\right] \\ &= \liminf_{n\to\infty} M_s^n \\ &= X_s \end{aligned}$$

Therefore X_t is a supermartingale.

Chapter 4

SDEs, Feynman-Kac and Generalised Black-Scholes

4.1 Stochastic Differential Equations

Definition 4.1.1: A (1-dim) SDE

A (1-d) stochastic differential equation is an equation of the form

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t$$

where $\mu : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are the drift and diffusion functions (note they do not depend on ω directly; randomness only arises through X_t), and W is a (1-d) standard Brownian motion. If the drift and diffusion does not depend on time, the SDE is called *time-homogenous or autonomous*.

Given an intial condition X_0, the solution to the SDE is a stochastic process X, that we call a *diffusion process*.

Example.

> **Geometric Brownian Motion (GBM).** Consider the SDE:
>
> $$\mathrm{d}X_t = \mu X_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t$$
>
> with $\mu, \sigma \in \mathbb{R}$. We have already shown the solution is the GBM. With initial condition X_0, we have
>
> $$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

Example.

> **Ornstein-Uhlenbeck (OU) with no drift.** Consider
>
> $$\mathrm{d}X_t = -\kappa X_t + \sigma\,\mathrm{d}W_t$$

> with $\kappa, \sigma \in \mathbb{R}$. With initial condition X_0, the solution is
>
> $$X_t = e^{-rt}\left(X_0 + \int_0^t e^{rs}\sigma \, \mathrm{d}W_s\right)$$

Proof. □

4.1.1 Strong solution

Definition 4.1.2: Strong solution to SDE

A process X is a *strong* solution to an SDE with initial condition X_0 if

(i) X is $\left(\mathcal{F}_t^W\right)_t$-adapted (the filtration generated by the Brownian motion, W)

(ii) the integrals $\int_0^T \mu(t, X_t)\,\mathrm{d}t$ and $\int_0^T \sigma(t, X_t)\,\mathrm{d}W_t$ are well defined for all T, that is $\mu_t \in \mathcal{L}^1$, and $\sigma_t \in \mathcal{H}^2_{\text{loc}}$

(iii) for all $t \geq 0$,

$$X_t = X_0 + \int_0^t \mu(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s \quad \text{a.s.}$$

given that we have a **fixed** Brownian motion W.

There is another type of solution; the *weak* solution. We will consider only strong solution here as when we price derivatives, the Brownian motion is fixed by the stock price process, and we from now one refer to 'strong' solutions as just the solution.

We state without proof:

Lemma 4.1.3: Markov property of strong solution

The (strong) solution, X to an SDE is a Markov process, that is for any function f,

$$\mathbb{E}\left[f(X_T)|\mathcal{F}_t\right] = \mathbb{E}\left[f(X_T)|X_t\right]$$

There is no dependence on the past.

4.1.2 Uniqueness and Existence of solutions

Definition 4.1.4: Lipschitz

A function $f(t, x)$ satsifies the *Lipschitz condition* w.r.t. x if there exists some constant $L \in \mathbb{R}$ s.t. for any t, and for any $x, y \in \mathbb{R}$ we have

$$|f_t(x) - f_t(y)| < L\,|x - y|$$

Note a continuous and differentiable function is Lipschitz iff it has a bounded first derivative, $|f_x(t, x)| \leq L$ (trivial application of Mean Value Theorem).

Definition 4.1.5: Growth condition

A function $f(t, x)$ satisfies the growth condition w.r.t. x if there exists some constant G s.t. for any t and for any $x \in \mathbb{R}$,

$$|f_t(x)| < G(1 + |x|)$$

Theorem 4.1.6: Uniqueness and existence of a solution to an SDE

If μ and σ satisfy both (1) Lipschitz and (2) growth conditions, then **there exists a unique strong solution**, X, that solves the SDE with initial condition X_0.

In the above theorem:

- uniqueness means that two solution processes are indistinguishable.
- the stated conditions are sufficient for existence and uniqueness, they are not necessary

Remark.

> A weaker condition under which uniqueness is obtained is when μ is Lipschitz but σ satisfies
>
> $$|\sigma(t, x) - \sigma(t, y)| < h(|x - y|)$$
>
> for all t and (x, y), where h is a strictly increasing function with $h(0) = 0$ that satisfies: for any $\varepsilon > 0$
>
> $$\int_0^{\varepsilon} \frac{1}{h(u)^2} \, \mathrm{d}u$$
>
> In particular, $h(u) = u^{\alpha}$ for $\alpha \geq 1/2$ works - justifying the Cox-Ingersoll-Ross process for the short rate.

4.1.3 Examples of linear SDEs

Definition 4.1.7: (1-d) Ornstein-Uhlenbeck SDE

Consider the linear SDE

$$\mathrm{d}r_t = \kappa(m - r_t) \, \mathrm{d}t + \sigma \, \mathrm{d}W_t$$

where $\kappa, m, \sigma > 0$ are positive constants. The solution r_t is said to follow an *Ornstein-Uhlenbeck process*.

Note r is a mean-reverting process; when r deviates from m, the drift $\kappa(m - r)$ pulls it back to its anchor level.

Lemma 4.1.8: Solution to the Ornstein-Uhlenbeck SDE

The solution to the Ornstein-Uhlenbeck SDE is

$$r_t = e^{-\kappa t} r_0 + m(1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa(s-t)} \, \mathrm{d}W_s$$

Proof for Lemma

Let $f(t) = e^{\kappa t}$. Applying Itô product rule to $r_t f(t)$, we get

$$\begin{aligned} \mathrm{d}(r_t f(t)) &= r_t \,\mathrm{d}f + f(t)\,\mathrm{d}r_t + 0 \\ &= r_t \kappa e^{\kappa t}\,\mathrm{d}t + e^{\kappa t}\left[\kappa(m - r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\right] \\ &= e^{\kappa t}\kappa m\,\mathrm{d}t + e^{\kappa t}\sigma\,\mathrm{d}W_t \end{aligned}$$

Thus

$$r_t f(t) = r_0 f(0) + \int_0^t e^{\kappa s}\kappa m\,\mathrm{d}s + \sigma\int_0^t e^{\kappa s}\,\mathrm{d}W_s$$

and

$$\begin{aligned} r_t &= e^{-\kappa t} r_0 + e^{-\kappa t} m \overbrace{\int_0^t e^{\kappa s}\kappa\,\mathrm{d}s}^{(e^{\kappa t}-1)} + e^{-\kappa t}\sigma\int_0^t e^{\kappa s}\,\mathrm{d}W_s \\ &= e^{-\kappa t} r_0 + m(1 - e^{-\kappa t}) + \sigma\int_0^t e^{\kappa(s-t)}\,\mathrm{d}W_s \end{aligned}$$

More generally for $0 \le s \le t$, the solution is

$$r_t = e^{-\kappa(t-s)} r_s + m(1 - e^{-\kappa(t-s)}) + \sigma\int_s^t e^{\kappa(u-t)}\,\mathrm{d}W_u$$

Because the integrand is deterministic, the Itô integral $\int_0^t e^{ku}\,\mathrm{d}W_u$ is normally distributed, so r_t is a Gaussian process.

Property 4.1.9: Properties of the Ornstein-Uhlenbeck process

The conditional mean and variance of r_t, an Ornstein-Uhlenbeck process, are

$$\begin{aligned} \mathbb{E}\left[r_t|\mathcal{F}_s\right] &= e^{\kappa(t-s)} r_s + m(1 - e^{-\kappa(t-s)}) \\ \mathrm{var}\left(r_t|\mathcal{F}_s\right) &= \sigma^2 e^{-2\kappa t}\mathrm{var}\left(\int_s^t e^{\kappa u}\,\mathrm{d}W_u\right) = \frac{\sigma^2}{2k}\left(1 - e^{-2\kappa(t-s)}\right) \end{aligned}$$

Proof

4.1.4 Generalised Linear SDEs

Definition 4.1.10: Linear SDE

A (1-d) *linear SDE* is of the form

$$\mathrm{d}X_t = [a(t)X_t + b(t)]\,\mathrm{d}t + [c(t)X_t + d(t)]\,\mathrm{d}W_t$$

where $a(t), b(t), c(t), d(t)$ are known (bounded) deterministic functions. We call this a *linear* SDE as the drift and diffusion functions are linear in X.

This class of SDEs is important as we can obtain explicit solutions. If

- $c = 0$, the solution is a Gaussian process

- $b = d = 0$, the solution is a stochastic exponential

Theorem 4.1.11: General solution to linear SDEs

Consider a generalised linear SDE. The solution is

$$X_t = Y_t X_0 + Y_t \int_0^t (b(s) - c(s)d(s))Y_s^{-1}\,\mathrm{d}s + Y_t \int_0^t d(s)Y_s^{-1}\,\mathrm{d}W_s$$

where

$$Y_t = \exp\left(\int_0^t a(s) - \frac{c(s)^2}{2}\,\mathrm{d}s + \int_0^t c(s)\,\mathrm{d}W_s\right)$$

Proof. By Itô's lemma,

$$\mathrm{d}(Y_t^{-1}) = (-a(t) + c(t)^2)Y_t^{-1}\,\mathrm{d}t - c(t)Y_t^{-1}\,\mathrm{d}W_t$$

Then, the integration by parts formula implies

$$\begin{aligned}\mathrm{d}(Y_t^{-1}X_t) &= Y_t^{-1}\,\mathrm{d}X_t + X_t\,\mathrm{d}(Y_t^{-1}) + \mathrm{d}(Y_t^{-1})\,\mathrm{d}X_t \\ &= Y_t^{-1}[(a(t)X_t + b(t))\,\mathrm{d}t + (c(t)X_t + d(t))\,\mathrm{d}W_t] \\ &\quad + X_t[(-a(t) + c(t)^2)Y_t^{-1}\,\mathrm{d}t - c(t)Y_t^{-1}\,\mathrm{d}W_t] \\ &\quad - c(t)Y_t^{-1}(c(t)X_t + d(t))\,\mathrm{d}t \\ &= (b(t) - c(t)d(t))Y_t^{-1}\,\mathrm{d}t + d(t)Y_t^{-1}\,\mathrm{d}W_t\end{aligned}$$

In integral form, we have

$$Y_t^{-1}X_t = Z_0X_0 + \int_0^t (b(s) - c(s)d(s))Y_s^{-1}\,\mathrm{d}s + \int_0^t d(s)Y_s^{-1}\,\mathrm{d}W_s$$

This implies that X_t is given by

$$X_t = Y_t X_0 + Y_t \int_0^t (b(s) - c(s)d(s))Y_s^{-1}\,\mathrm{d}s + Y_t \int_0^t d(s)Y_s^{-1}\,\mathrm{d}W_s$$

□

4.2 From SDEs to PDEs

4.2.1 Feynman-Kac and the Black-Scholes PDE

Theorem 4.2.1: (Reverse) Feynman-Kac

For given functions ρ, g, μ, σ, let

$$f(t,x) = \mathbb{E}\left[e^{-\int_t^T \rho(u,X_u)\,\mathrm{d}u} g(X_T) | \mathcal{F}_t\right]$$

where X is the solution to the SDE with initial condition $X_0 = x$

$$\mathrm{d}X_u = \mu(u,X_u)\,\mathrm{d}u + \sigma(u,X_u)\,\mathrm{d}W_u$$

(Reverse Feynman-Kac) If f is defined above in $C^{1,2}$ and $(\sigma(t,X_t)f_x(t,X_t)) \in \mathcal{H}^2$, then it is the solution to the following PDE:

$$f_t(t,x) + \mu(t,x)f_x(t,x) + \frac{1}{2}\sigma^2(t,x)f_{xx}(t,x) - \rho(t,x)f(t,x) = 0$$

and satisfies terminal condition $f(T,x) = g(x)$.

Proof. Let X be the solution to the SDE starting at time 0. We have

$$f(t,X_t) = \mathbb{E}\left[e^{-\int_t^T \rho(u,X_u)\,\mathrm{d}u} g(X_T) | \mathcal{F}_t\right]$$

Define Y_t as the product of f and an exponential

$$Y_t = e^{-\int_0^t \rho(u,X_u)\,\mathrm{d}u} f(t,X_t) = \mathbb{E}\left[e^{-\int_0^T \rho(u,X_u)\,\mathrm{d}u} g(X_T) | \mathcal{F}_t\right]$$

Importantly, note that Y_t is a martingale (by Law of Iterated Expectations) $\implies$ drift $= 0$

By the Itô product rule,

$$\mathrm{d}Y_t = f(t,X_t)\,\mathrm{d}(e^{-\int_0^t \rho(u,X_u)\,\mathrm{d}u}) + e^{-\int_0^t \rho(u,X_u)\,\mathrm{d}u}\,\mathrm{d}f(t,X_t) + 0$$

We have by Itô's lemma on f,

$$\begin{aligned}
&\mathrm{d}(e^{-\int_0^t \rho(u,X_u)\,\mathrm{d}u}) = -\rho(t,X_t)e^{-\int_0^t \rho(u,X_u)\,\mathrm{d}u}\,\mathrm{d}t \\
&\mathrm{d}f(t,X_t) = f_t(t,X_t)\,\mathrm{d}t + \mu(t,X_t)f_x(t,X_t)\,\mathrm{d}t + \sigma(t,X_t)f_x(t,X_t)\,\mathrm{d}W_t + \frac{\sigma(t,X_t)^2}{2}f_{xx}(t,X_t)\,\mathrm{d}t
\end{aligned}$$

thus $\mathrm{d}Y_t$ becomes

$$\begin{aligned}
\mathrm{d}Y_t = -f(t,X_t)\left[\rho(t,X_t)e^{-\int_0^t \rho(u,X_u)\,\mathrm{d}u}\,\mathrm{d}t\right] + e^{-\int_0^t \rho(u,X_u)\,\mathrm{d}u}\Bigg[f_t(t,X_t)\,\mathrm{d}t \\
+ \underbrace{\mu(t,X_t)f_x(t,X_t)\,\mathrm{d}t + \sigma(t,X_t)f_x(t,X_t)\,\mathrm{d}W_t}_{=f_x(t,X_t)\,\mathrm{d}X_t} + \frac{\sigma(t,X_t)^2}{2}f_{xx}(t,X_t)\,\mathrm{d}t\Bigg]
\end{aligned}$$

Collecting $\mathrm{d}t$ and $\mathrm{d}W_t$ terms,

$$\mathrm{d}Y_t = e^{-\int_0^t \rho(u,X_u)\,\mathrm{d}u}\Bigg[-\rho(t,X_t)f(t,X_t) + f_t(t,X_t) + \mu(t,X_t)f_x(t,X_t)$$

$$+ \frac{\sigma(t,X_t)^2}{2} f_{xx}(t,X_t)\Bigg] \,\mathrm{d}t + e^{-\int_0^t \rho(u,X_u)\,\mathrm{d}u} \sigma(t,X_t) f_x(t,X_t)\,\mathrm{d}W_t$$

The drift is zero, therefore we obtain the PDE in the theorem.

The terminal condition, $f(T, X_T) = \mathbb{E}_T\left[g(X_T)\right] = g(X_T)$ □

In finance, risk-neutral valuation typically gives us expressions for security prices which look like function f. Transforming the computation of an expectation into a Cauchy problem is useful when there is no tractable direct way to compute the expectation. In principle, Monte-Carlo methods allow us to compute the expectation for each (t, x) by simulation of the SDE. But solving numerically the Cauchy problem (e.g., by finite-difference method) is often more efficient.

Corollary 4.2.2: Black-Scholes PDE

Consider an asset with payoff $F(S_T)$, from the previous chapter we have shown that by no-arbitrage, we have

$$P_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-rt}F(S_T)|\mathcal{F}_t\right]$$

Then by the Feynman-Kac theorem, we have that the price, satisfies the Black-Scholes PDE

$$f_t(t,s) + rsf_s(t,s) + \frac{1}{2}\sigma^2 s^2 f_{ss}(t,s) - rf(t,s) = 0$$

with terminal condition $f(T, S_T) = F(S_T)$ and

$$P_t = f(t, S_t)$$

Proof for Corollary.

Trivial application of Feynman-Kac under EMM.

4.2.2 Pricing Exotic Derivatives

Definition 4.2.3: Down-and-out Barrier Option

An option that yields a payoff

$$F(S_T)\mathbf{1}_{\{\underline{S}_T > b\}}$$

at maturity time $T > 0$, where $b < S_0$ is the knockout barrier and $\underline{S}_T := \min_{u\in[0,t]} S_u$ is called a **down-and-out barrier option**.

Definition 4.2.4: Down-and-in Barrier Option

An option that yields a payoff

$$F(S_T)\mathbf{1}_{\{\underline{S}_T < b\}}$$

at maturity time $T > 0$, where $b < S_0$ is the knockout barrier and $\underline{S}_T := \min_{u\in[0,t]} S_u$ is called a **down-and-in barrier option**.

Theorem 4.2.5: Pricing Barrier Options

Suppose the function $(t,s) \to v(t,s)$ is $C^{1,2}$ and satisfies the Black-Scholes PDE

$$v_t(t,s) + rsv_s(t,s) + \frac{1}{2}\sigma^2 s^2 v_{ss}(t,s) - rv(t,s) = 0$$

with terminal conditions

$$\begin{aligned} v(t,b) &= 0, \quad \text{for } t \in [0,T] \\ v(T,s) &= F(s), \quad \text{for } s > b \end{aligned}$$

Consider an option with payoff $F(S_T)\mathbf{1}_{\{\underline{S}_T > b\}}$. The n*o-arbitrage price of the option* is

$$P_t = v(t,S_t)\mathbf{1}_{\{\underline{S}_t > b\}}$$

and the *associated hedging process* is given by

$$\pi_t^* = S_t \,\mathrm{d}v_s(t,S_t)\mathbf{1}_{\{\underline{S}_t > b\}}$$

Proof. We start with:

Step 1. Consider the $(\mathcal{F}_t)$-stopping time τ defined by

$$\tau := \inf\{t \geq 0 : S_t \leq b\}$$

i.e. the shortest time that S_t falls to or below b. Note that this means

$$\{\omega \in \Omega : \underline{S}_t > b\} = \{\omega \in \Omega : \tau(\omega) > t\}$$

which implies

$$\mathbf{1}_{\{\underline{S}_t > b\}} = \mathbf{1}_{\{\tau > t\}}$$

Step 2. Consider the *discounted value process*

$$\begin{aligned} \mathrm{d}(e^{-rt}v(t,S_t)) &= v(t,S_t)\,\mathrm{d}(e^{-rt}) + e^{-rt}\,\mathrm{d}v(t,S_t) + 0 \\ &= -re^{-rt}v(t,S_t)\,\mathrm{d}t + e^{-rt}\left(v_t(t,S_t)\,\mathrm{d}t + v_s(t,S_t)\,\mathrm{d}S_t + \frac{1}{2}v_{ss}(t,S_t)\,\mathrm{d}\langle S\rangle_t\right) \\ &= -re^{-rt}v(t,S_t)\,\mathrm{d}t + e^{-rt}\big[v_t(t,S_t)\,\mathrm{d}t \\ &\qquad + v_s(t,S_t)\left(rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\vartheta}\right) + \frac{1}{2}\sigma^2 S_t^2 v_{ss}(t,S_t)\,\mathrm{d}t\big] \\ &= e^{-rt}\overbrace{\left[v_t(t,s) + rsv_s(t,s) + \frac{1}{2}\sigma^2 s^2 v_{ss}(t,s) - rv(t,s)\right]}^{=0}\mathrm{d}t \\ &\qquad + e^{-rt}\sigma S_t v_s(t,S_t)\,\mathrm{d}W_t^{\vartheta} \end{aligned}$$

In integral form, the value process is

$$e^{-r(t\wedge\tau)}v(t\wedge\tau, S_{t\wedge\tau}) = v(0,S_0) + \int_0^{t\wedge\tau} e^{-ru}\sigma S_u v_s(u,S_u)\,\mathrm{d}W_u^{\vartheta}$$

Step 3. Recall the general *discounted portfolio process*:

$$\mathrm{d}(e^{-rt}X_t) = e^{-rt}\pi_t\sigma\,\mathrm{d}W_t^{\vartheta}$$

In integral form, we have

$$e^{-r(t\wedge\tau)}X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} e^{-ru}\pi_u\sigma\,\mathrm{d}W_u^{\vartheta}$$

More specifically, the wealth process, X^* for a portfolio with initial wealth $v(0,S_0)$ and position $\pi_t^* = S_t v_s(t,S_t)\mathbf{1}_{\{\tau>t\}}$ is

$$e^{-r(t\wedge\tau)}X^*_{t\wedge\tau} = v(0,S_0) + \int_0^{t\wedge\tau} e^{-ru}S_u v_s(u,S_u)\mathbf{1}_{\{\tau>u\}}\sigma\,\mathrm{d}W_u^{\vartheta}$$

It holds for all t,τ that $t\wedge\tau \le \tau$, thus we can drop the indicator. By looking at their respective processes, we have that

$$X^*_{t\wedge\tau} = v(t\wedge\tau, S_{t\wedge\tau}), \quad \forall t\in[0,T]$$

Step 4. Now we check boundary conditions; i.e. what happens at τ and T?

$$\begin{aligned} X^*_\tau = X^*_{t\wedge\tau}\mathbf{1}_{\{\tau\le t\}} &= v(t\wedge\tau, S_{t\wedge\tau})\mathbf{1}_{\{\tau\le t\}} \\ &= v(\tau, S_\tau)\mathbf{1}_{\{\tau\le t\}} \\ &= v(\tau, b)\mathbf{1}_{\{\tau\le t\}} \\ &= 0 \end{aligned}$$

Using this τ boundary condition we have that for T

$$\begin{aligned} X^*_T &= X^*_T\mathbf{1}_{\{\tau\le T\}} + X^*_T\mathbf{1}_{\{\tau>T\}} \\ &= X^*_{t\wedge\tau}\mathbf{1}_{\{\tau\le T\}} + X^*_{t\wedge\tau}\mathbf{1}_{\{\tau>T\}} \\ &= 0 + v(T\wedge\tau, S_{T\wedge\tau})\mathbf{1}_{\{\tau>T\}} \\ &= v(T, S_T)\mathbf{1}_{\{\tau>T\}} \\ &= F(S_T)\mathbf{1}_{\{\tau>T\}} \end{aligned}$$

which matches the option payoff.

□

Definition 4.2.6: Asian Call Option

An asset that has payoff

$$\left(\frac{1}{T}\int_0^T S_u\,\mathrm{d}u - K\right)^+$$

with maturity $T>0$, and strike $K>0$ is called an Asian call option.

Remark.

> For notational ease, let us define
>
> $$Y_t := \frac{1}{S_t}\left(\frac{1}{T}\int_0^t S_u \,\mathrm{d}u - K\right)$$
>
> then the payoff of the Asian call option is
>
> $$S_T(Y_T)^+$$

Theorem 4.2.7: Pricing Asian Options

Suppose the function $(t,y) \to g(t,y)$ is a $C^{1,2}$ solution to the PDE

$$g_t(t,y) + \frac{1}{2}\sigma^2 y^2 g_{yy}(t,y) + \left(\frac{1}{T} - ry\right) g_y(t,y) = 0 \tag{4.2.1}$$

with terminal conditions

$$g(T,y) = y^+, \quad \text{for } y \in \mathbb{R}$$

Then the no-arbitrage price of the option is given by

$$P_t = S_t g(t, Y_t), \quad \text{for } t \in [0,T]$$

Proof. We start with:

Step 1. Some preliminaries;

- Let $f(x) = 1/x$, then $f' = -1/x^2$, $f'' = 2/x^3$. By Itô's lemma,

$$\begin{aligned}\mathrm{d}S_t^{-1} = \mathrm{d}f(S_t) &= f'(S_t)\,\mathrm{d}S_t + \frac{1}{2}f''(S_t)\,\mathrm{d}\langle S\rangle_t \\ &= -\frac{1}{S_t^2}(rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^\vartheta) + \frac{1}{S_t^3}\sigma^2 S_t^2\,\mathrm{d}t \\ &= \frac{\sigma^2 - r}{S_t}\,\mathrm{d}t - \frac{\sigma}{S_t}\,\mathrm{d}W_t^\vartheta\end{aligned}$$

- Define $Z_t := \frac{1}{T}\int_0^t S_u\,\mathrm{d}u - K$, thus

  $$\mathrm{d}Z_t = \frac{1}{T}S_t\,\mathrm{d}t$$

  and $Y_t = f(S_t)Z_t$
- By Itô product rule, the evolution of Y_t is

$$\begin{aligned}\mathrm{d}Y_t &= f(S_t)\,\mathrm{d}Z_t + Z_t\,\mathrm{d}f(S_t) + 0 \\ &= \frac{1}{S_t}\cdot\frac{1}{T}S_t\,\mathrm{d}t + Z_t\left(\frac{\sigma^2 - r}{S_t}\,\mathrm{d}t - \frac{\sigma}{S_t}\,\mathrm{d}W_t^\vartheta\right) \\ &= \left(\frac{1}{T} + (\sigma^2 - r)Y_t\right)\mathrm{d}t - \sigma Y_t\,\mathrm{d}W_t^\vartheta\end{aligned}$$

- By Itô's lemma, the evolution of $g(t, Y_t)$ is

$$\begin{aligned}
\mathrm{d}g(t,Y_t) &= g_t(t,Y_t)\,\mathrm{d}t + g_y(t,Y_t)\,\mathrm{d}Y_t + \frac{1}{2}g_{yy}(t,Y_t)\,\mathrm{d}\langle Y\rangle_t \\
&= g_t(t,Y_t)\,\mathrm{d}t + g_y(t,Y_t)\left[\left(\frac{1}{T} + (\sigma^2 - r)Y_t\right)\mathrm{d}t - \sigma Y_t\,\mathrm{d}W_t^\vartheta\right] + \frac{1}{2}g_{yy}(t,Y_t)\sigma^2 Y_t^2\,\mathrm{d}t \\
&= \left[g_t(t,Y_t) + g_y(t,Y_t)\left(\frac{1}{T} + (\sigma^2 - r)Y_t\right) + \frac{1}{2}g_{yy}(t,Y_t)\sigma^2 Y_t^2\right]\mathrm{d}t - g_y(t,Y_t)\sigma Y_t\,\mathrm{d}W_t^\vartheta \\
&= \overbrace{\left[g_t + g_y\left(\frac{1}{T} - rY_t\right) + \frac{1}{2}g_{yy}\sigma^2 Y_t^2\right]}^{=0}\mathrm{d}t + g_y\sigma^2 Y_t\,\mathrm{d}t - g_y\sigma Y_t\,\mathrm{d}W_t^\vartheta \\
&= g_y\sigma^2 Y_t\,\mathrm{d}t - g_y\sigma Y_t\,\mathrm{d}W_t^\vartheta
\end{aligned}$$

Step 2. Recall the discounted stock price process is $\mathrm{d}(e^{-rt}S_t) = e^{-rt}\sigma S_t\,\mathrm{d}W_t^\vartheta$, therefore the discounted price process by Itô product rule is

$$\begin{aligned}
\mathrm{d}(e^{-rt}S_t g(t,Y_t)) &= e^{-rt}S_t\,\mathrm{d}g(t,Y_t) + g(t,Y_t)\,\mathrm{d}(e^{-rt}S_t) - e^{-rt}g_y(t,Y_t)\sigma^2 Y_t S_t\,\mathrm{d}t \\
&= e^{-rt}S_t\left[g_y\sigma^2 Y_t\,\mathrm{d}t - g_y\sigma Y_t\,\mathrm{d}W_t^\vartheta\right] + e^{-rt}g\sigma S_t\,\mathrm{d}W_t^\vartheta - e^{-rt}g_y\sigma^2 Y_t S_t\,\mathrm{d}t \\
&= -e^{-rt}S_t g_y\sigma Y_t\,\mathrm{d}W_t^\vartheta + e^{-rt}g\sigma S_t\,\mathrm{d}W_t^\vartheta \\
&= e^{-rt}\sigma S_t\left(g - g_y Y_t\right)\mathrm{d}W_t^\vartheta
\end{aligned}$$

In integral form,

$$e^{-rt}S_t g(t,Y_t) = S_0 g(0,Y_0) + \int_0^t S_u\left(g - g_y Y_u\right)\mathrm{d}W_u^\vartheta$$

Step 3. On the other hand, the discounted wealth process for π_t^* that starts with initial capital $X_0 = S_0 g(0,Y_0)$ satisfies

$$\begin{aligned}
e^{-rt}X_t &= S_0 g(0,Y_0)\int_0^t e^{-ru}\pi_u^*\sigma\,\mathrm{d}W_u^\vartheta \\
&= S_0 g(0,Y_0) + \int_0^t e^{-ru}S_u(g(u,Y_u) - Y_u g_y(u,Y_u))\sigma\,\mathrm{d}W_u^\vartheta
\end{aligned}$$

which corresponds to the discounted price process. Therefore

$$X_t = S_t g(t,Y_t) \quad \mathbb{P}\text{-a.s.}$$

Step 4. Check boundary condition, X_T is

$$\begin{aligned}
X_T &= S_T g(T,Y_T) \\
&= S_T(Y_T)^+
\end{aligned}$$

which matches the option payoff.

□

4.3 Generalised Black-Scholes

Definition 4.3.1: Generalised Black-Scholes Market

Assume the price of the stock follows

$$\frac{\mathrm{d}S_t}{S_t} = \mu(t, S_t)\,\mathrm{d}t + \sigma(t, S_t)\,\mathrm{d}W_t$$

and the money market follows

$$\mathrm{d}B_t = r(t, S_t)B_t\,\mathrm{d}t$$

We assume:

- μ and σ are **deterministic functions** s.t. there is a unique strong solution to the stock price process, and the solution is Markov.
- σ is bounded away from zero
- r is integrable and is bounded from below
- the process L defined as

$$L_t := \exp\left(-\int_0^t \frac{\mu(u, S_u) - r(u, S_u)}{\sigma(u, S_u)}\,\mathrm{d}W_t - \frac{1}{2}\int_0^t \left(\frac{\mu(u, S_u) - r(u, S_u)}{\sigma(u, S_u)}\right)^2 \mathrm{d}u\right)$$

  is **well-defined** and a **martingale**.

This model generalises the Black-Scholes framework. We call the deterministic function

$$(t, s) \to \sigma(t, s)$$

local volatility. Why do we specify these assumptions?

- Pricing and hedging of contingent claims is basically the same as in Black-Scholes.
- Market is complete (all contingent claims can be replicated by trading in the underlying) and admits no arbitrage.

As in the last chapter, we will prove the similar properties to the standard Black-Scholes market after a brief discussion on local volatility.

4.3.1 Local volatility

Some specifications often used for local volatility, however note it remains to be checked that the L given is a martingale.

Definition 4.3.2: Constant elasticity of variance (CEV) model

The *constant elasticity of variance* model specifies

$$\sigma(t, s) = \delta s^{\beta}$$

for the local volatility function, where $\delta > 0$ and usually $\beta \leq 0$

Example.

> We look at the various cases for β.
>
> - $\beta > 0$ needs care...
> - $\beta = -\frac{1}{2}$ is also known as the Cox-Ingersoll-Ross (CIR) or square-root process, often used for modelling the short rate
> - $\beta < 0$ yields a **leverage effect** where spot volatility increases as asset price declines (phenomenon observed in data)
>
> In general the rule is that we need to be cautious about whether there is a positive probability that the stock price hitting zero. In addition, processes are often chosen for tractability purposes.
>
> The CEV model can be extended for **stochastic** δ, called the SABR model (stochastic alpha, beta, rho).

Definition 4.3.3: Quadratic normal volatility model

If we define risk-neutral stock price dynamics as

$$\mathrm{d}S_T = r(t, S_t)S_t + (aS_t^2 + bS_t + c)\,\mathrm{d}W_t^{\vartheta}$$

this implies the local volatility function is given by

$$\sigma(t, s) = as + b + \frac{c}{s}$$

Quadratic normal volatility is often used to price FX options. Strict local martingality is again an issue, but process is analytically tractable.

4.3.2 Self-financing portfolios

Consider a dynamic self-financing portfolio strategy where $\pi_t := N_t S_t$ denotes holdings (in money) of the risky asset. Note π is *adapted*, that is the position at time t can only depend on the information available at time t.

Theorem 4.3.4: Self-financing portfolio SDE

The value of the self-financing portfolio follows an SDE

$$\mathrm{d}X_t = r(t, S_t)X_t\,\mathrm{d}t + [\mu(t, S_t) - r(t, S_t)]\pi_t\,\mathrm{d}t + \sigma(t, S_t)\pi_t\,\mathrm{d}W_t$$

Proof. Let (N^0, N) be units of risk-free and risky assets, respectively. The portfolio's total value X_t at time t is

$$X_t = N_t^0 B_t + N_t S_t$$

and, since it is self-financing, $X_{t-} = X_{t+}$.

Investor rearranges the portfolio's positions:

- at discrete times only, at no transaction costs

- portfolio's positions change at times t and $t+\Delta$ but not at any intermediate time

$$X_{t+\Delta} - X_t = N_t^0 (B_{t+\Delta} - B_t) - N_t (S_{t+\Delta} - S_t)$$

In continuous time we have

$$\begin{aligned} \mathrm{d}X_t &= N_t^0 \,\mathrm{d}B_t + N_t \,\mathrm{d}S_t \\ &= \left[r(t,S_t) N_t^0 B_t + \mu(t,S_t) N_t S_t \right] \mathrm{d}t + \sigma(t,S_t) N_t S_t \,\mathrm{d}W_t \\ &= r(t,S_t) X_t \,\mathrm{d}t + \left[\mu(t,S_t) - r(t,S_t) \right] \pi_t \,\mathrm{d}t + \sigma(t,S_t) \pi_t \,\mathrm{d}W_t \end{aligned}$$

In integral form, we have

$$X_t = X_0 + \int_0^t r(u,S_u) X_u \,\mathrm{d}u + \int_0^t \left[\mu(u,S_u) - r(u,S_u) \right] \pi_t \,\mathrm{d}u + \int_0^t \sigma(u,S_u) \pi_u \,\mathrm{d}W_u$$

□

4.3.3 Girsanov revisited

Almost exactly as before in the standard Black-Scholes model.

Definition 4.3.5: Price of risk

We define

$$\vartheta(t,S_t) = \frac{\mu(t,S_t) - r(t,S_t)}{\sigma(t,S_t)}$$

as the *price of risk*.

Consider the exponential martingale L given by

$$\mathrm{d}L_t = -\vartheta(t,S_t) L_t \,\mathrm{d}W_t; \quad L_0 = 1$$

The solution is

$$L_t = \exp\left(-\frac{1}{2} \int_0^t \vartheta^2 \,\mathrm{d}s - \int_0^t \vartheta_s \,\mathrm{d}W_s \right)$$

Definition 4.3.6: Equivalent Martingale Measure

The *risk-neutral* or *equivalent martingale measure*, $\mathbb{Q}_T$ on measurable space $(\Omega, \mathcal{F}_T)$ is defined by

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}} \left[L_T \mathbf{1}_{\{A\}} \right], \quad \text{for } A \in \mathcal{F}_T$$

In this case, we take as given Girsanov's theorem (proof is even more involved). Note that we have an integral in the definition of W_t^{ϑ}.

Theorem 4.3.7: Girsanov's Theorem

Under the measure $\mathbb{Q}$, the process $(W_t^\vartheta)_{t\in[0,T]}$ given by

$$W_t^\vartheta = \int_0^t \vartheta(u, S_u)\,\mathrm{d}u + W_t$$

is an $(\mathcal{F}_t)$-Brownian motion.

Corollary 4.3.8

For any $Z \in \mathcal{F}_T$ and $s \leq T$ we have

$$\mathbb{E}^{\mathbb{Q}}\left[Z|F_s\right] = \frac{\mathbb{E}^{\mathbb{P}}\left[L_T Z|\mathcal{F}_s\right]}{L_s}$$

4.3.4 Admissibility and Arbitrage

Theorem 4.3.9: Local Martingale property

In the generalised Black-Scholes market,

1. the **discounted stock price process**,

$$\left(e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} S_t\right)_{t\in[0,T]}$$

2. the **discounted portfolio wealth process**,

$$\left(e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} X_t\right)_{t\in[0,T]}$$

are $(\mathcal{F}_t)$-*local martingales* under the measure $\mathbb{Q}$.

Proof. We consider the general self-financing portfolio. Integrating by parts

$$\begin{aligned}
\mathrm{d}\left(e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} X_t\right) &= X_t\,\mathrm{d}\left(e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\right) + e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\,\mathrm{d}X_t + 0 \\
&= -X_t r(t,S_t)\left(e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\right)\mathrm{d}t \\
&\quad + e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\left[r(t,S_t)X_t\,\mathrm{d}t + \left(\mu(t,S_t) - r(t,S_t)\right)\pi_t\,\mathrm{d}t + \sigma(t,S_t)\pi_t\,\mathrm{d}W_t\right] \\
&= e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\left[\left(\mu(t,S_t) - r(t,S_t)\right)\pi_t\,\mathrm{d}t + \sigma(t,S_t)\pi_t\,\mathrm{d}W_t\right] \\
&= e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\sigma(t,S_t)\pi_t\left[\frac{\mu(t,S_t) - r(t,S_t)}{\sigma(t,S_t)}\,\mathrm{d}t + \mathrm{d}W_t\right]
\end{aligned}$$

Moreover, by Girsanov's Theorem,

$$\mathrm{d}\left(e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} X_t\right) = e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\sigma(t,S_t)\pi_t\,\mathrm{d}W_t^\vartheta$$

Thus we have that $\left(e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} X_t\right)$ is an $(\mathcal{F}_t)$-local martingale under $\mathbb{Q}$ (necessary condition for Itô integrability) ☐

Theorem 4.3.10: Admissible $\implies$ no-arbitrage

In the **generalised** Black-Scholes market, *there is no admissible self-financing portfolio offering arbitrage opportunities.*

Proof. Assume $\exists\ \pi$, i.e. an admissible portfolio process π with initial endowment $X_0 = 0$, s.t. the associated self-financing portfolio wealth X satisfying an arbitrage

- for some $T > 0$, $\mathbb{P}\left(X_T \geq 0\right) = 1$
- for some $T > 0$, $\mathbb{P}\left(X_T \geq 0\right) > 0$

As the risk-neutral measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$, thus

- for some $T > 0$, $\mathbb{Q}\left(X_T \geq 0\right) = 1$
- for some $T > 0$, $\mathbb{Q}\left(X_T \geq 0\right) > 0$

We proceed now with the contradiction:

Step 1. **(Conditions imposed by admissibility)** Since the portfolio strategy π is admissible, there exists a constant $\tilde{K} \geq 0$ s.t. $X_T \geq -\tilde{K}$. Since r is bounded from below, there exists a constant $R > 0$ s.t. $r(t, s) \geq -R$ for all $t, s > 0$, thus $e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} \leq e^{Rt} < \infty$.

Combining these two, for all $t \in [0, T]$ we have

$$X_t e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} \geq -\tilde{K} e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} \geq -\tilde{K} e^{Rt} =: -K > -\infty$$

Thus $X_t e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} \geq -K \implies X_t e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} + K \geq 0$ is a positive $(\mathcal{F}_t)$-local martingale with respect to the probability measure $\mathbb{Q}$, and is therefore a **supermartingale**, that is

$$\mathbb{E}^{\mathbb{Q}}\left[X_t e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} + K | \mathcal{F}_0\right] \leq X_0 + K = K$$

In other words,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(u,S_u)\,\mathrm{d}u} X_T\right] &= \left(\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(u,S_u)\,\mathrm{d}u} X_T + K\right] - K\right) \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} X_T + K | \mathcal{F}_0\right]\right] - K \\ &\leq \mathbb{E}^{\mathbb{Q}}\left[X_0 + K\right] - K \\ &= 0 \end{aligned}$$

Step 2. **(Conditions imposed by arbitrage existence)** As $\int_0^T |r(u, S_u)|\,\mathrm{d}u < \infty$, we have

$$X_T > 0 \iff e^{-\int_0^T r(u,S_u)\,\mathrm{d}u} X_T > 0$$

Since we assume an arbitrage exists under $\mathbb{Q}$, we have

$$\mathbb{Q}\left(e^{-\int_0^T r(u,S_u)\,\mathrm{d}u} X_T \geq 0\right) = 1 \quad \text{and} \quad \mathbb{Q}\left(e^{-\int_0^T r(u,S_u)\,\mathrm{d}u} X_T > 0\right) = 1$$

which implies

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(u,S_u)\,\mathrm{d}u} X_T\right] > 0$$

Since admissibility and arbitrage independently produce contradicting results, the two cannot both be true. □

4.3.5 Valuation

Theorem 4.3.11: Existence and pricing of European claims

Consider a European contingent claim with payoff $H_T \in \mathcal{L}^1(\Omega, \mathcal{F}_T, \mathbb{Q})$. There exists a portfolio s.t. $X_T = H_T$. Moreover the time t value of such a portfolio is given by

$$X_t = \mathbb{E}^{\mathbb{P}}\left[\frac{B_t}{B_T}\frac{L_T}{L_t}H_T|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_T}H_T|\mathcal{F}_t\right], \quad \text{for } t \in [0, T]$$

Motivation for Proof. The proof is essentially same as before. If we can find a replicating portfolio π_t s.t. the associated wealth process

$$X_T(X_0, \pi) = H_T$$

then the value of the portfolio is the same as the asset H_t at time $t \leq T$, that is

$$B_t^{-1}X_t^{\pi} = \frac{1}{L_t}\mathbb{E}^{\mathbb{P}}\left[B_T^{-1}L_T H_T|\mathcal{F}_t\right]$$

where $B_t := e^{\int_0^t r(u,S_u)\,\mathrm{d}u}$. Now consider the $(\mathcal{F}_t, \mathbb{P})$-martingale M defined by

$$M_t := \mathbb{E}^{\mathbb{P}}\left[B_T^{-1}L_T H_T|\mathcal{F}_t\right]$$

s.t. $B_t^{-1}X_t = \frac{M_t}{L_t}$. Then by the martingale representation theorem, there exists a K that is $(\mathcal{F}_t)$-adapted s.t. for all $t \geq 0$

$$M_t = \mathbb{E}^{\mathbb{P}}\left[B_T^{-1}L_T H_T\right] + \int_0^t K_s\,\mathrm{d}W_s \iff \mathrm{d}M_t = K_t\,\mathrm{d}W_t = K_t(\mathrm{d}W_t^{\vartheta} - \theta\,\mathrm{d}t)$$

Now recall that $\mathrm{d}L_t = -\theta L_t\,\mathrm{d}W_t$, which by Itô's lemma implies

$$\mathrm{d}L_t^{-1} = \theta^2 L_t^{-1}\,\mathrm{d}t + \theta L_t^{-1}\,\mathrm{d}W_t = \theta L_t^{-1}\,\mathrm{d}W_t^{\vartheta}$$

Integrating by parts, we have

$$\begin{aligned}\mathrm{d}\left(\frac{M_t}{L_t}\right) &= M_t\,\mathrm{d}L_t^{-1} + L_t^{-1}\,\mathrm{d}M_t + \theta K_t L_t^{-1}\,\mathrm{d}t \\ &= M_t\theta L_t^{-1}\,\mathrm{d}W_t^{\vartheta} + L_t^{-1}K_t\,\mathrm{d}W_t^{\vartheta} - L_t^{-1}\theta K_t\,\mathrm{d}t + \theta K_t L_t^{-1}\,\mathrm{d}t \\ &= (M_t\theta + K_t)L_t^{-1}\,\mathrm{d}W_t^{\vartheta}\end{aligned}$$

and

$$\frac{M_t}{L_t} = \frac{M_0}{L_0} + \int_0^t \frac{M_s\theta + K_s}{L_s}\,\mathrm{d}W_s^{\vartheta}$$

Proof. Consider the self-financing strategy π_t with initial wealth

$$X_0 = M_0 \qquad \text{(which is also just } \frac{M_0}{L_0}\text{)}$$

and process defined by

$$\pi_t := B_t\left(\frac{M_t\theta + K_t}{\sigma(t, S_t)L_t}\right)$$

Therefore using the portfolio evolution definition, the discounted wealth process can be written as

$$\begin{aligned} B_t^{-1}X_t &= X_0 + \int_0^t B_t^{-1}\pi_u\sigma(u,S_u)\,\mathrm{d}W_u^\vartheta \\ &= \frac{M_0}{L_0} + \int_0^t \frac{M_s\theta + K_s}{L_s}\,\mathrm{d}W_s^\vartheta \end{aligned}$$

We showed above that the RHS can be expressed simply as M_t/L_t through repeated application of Itô calculus. Therefore at time T, we have that X_T by our definition of M_t is

$$\begin{aligned} X_T &= B_T\frac{M_T}{L_T} \\ &= \frac{B_T}{L_T}\mathbb{E}^{\mathbb{Q}}\left[B_T^{-1}L_TH_T|\mathcal{F}_T\right] \\ &= H_T \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Therefore since there exists a replicating portfolio, then the price at time t, H_t must be equal to the wealth process at time t, X_t. ☐

Corollary 4.3.12: No-arbitrage price

Consider a European contingent claim with payoff H_T at maturity $T > 0$, and suppose there exists a self-financing portfolio strategy π with initial value X_0 s.t. the associated wealth process satisfies

$$X_T = H_T \quad \mathbb{P}\text{-a.s.}$$

Then the *no-arbitrage price*, P_t of the claim is

$$P_t = X_t, \quad \forall t \in [0,T]$$

On the other hand, we also have the PDE approach like before. We show that if a function f satisfies the generalised BS PDE, then the price, $P_t = f(t, S_t)$.

Theorem 4.3.13: Generalised Black-Scholes PDE

Consider an option with payoff $F(S_T)$. Suppose that the function $(t,y) \to f(t,y)$ is a $C^{1,2}$ solution to the PDE

$$f_t(t,s) + \frac{1}{2}\sigma^2(t,s)s^2f_{ss}(t,s) + r(t,s)sf_s(t,s) - r(t,s)f(t,s) = 0$$

with boundary condition

$$f(T,s) = F(s), \quad \text{for } y \in \mathbb{R}$$

The no-arbitrage price of the option is

$$X_t^* = f(t,S_t), \quad \text{for } t \in [0,T]$$

and the associated *hedging portfolio process* is given by

$$\pi_t^* = S_tf_s(t,S_t), \quad \text{for } t \in [0,T]$$

Proof. Consider discounted price $f(t,S_t)B_t^{-1}$ where $B_t^{-1} = e^{-\int_0^t ru,S_u)\,\mathrm{d}u}$. As $\mathrm{d}B_t^{-1} = -r(t,S_t)B_t^{-1}\,\mathrm{d}t$, integrating by parts yields

$$\begin{aligned}
\mathrm{d}\big(f(t,S_t)B_t^{-1}\big) &= f(t,S_t)\,\mathrm{d}B_t^{-1} + B_t^{-1}\,\mathrm{d}f(t,S_t) + 0 \\
&= -r(t,S_t)f(t,S_t)B_t^{-1}\,\mathrm{d}t + B_t^{-1}\left[f_t(t,S_t)\,\mathrm{d}t + f_s(t,S_t)\,\mathrm{d}S_t + \frac{1}{2}f_{ss}(t,S_t)\,\mathrm{d}\langle S\rangle_t\right] \\
&= -r(t,S_t)f(t,S_t)B_t^{-1}\,\mathrm{d}t \\
&\quad + B_t^{-1}\left[f_t(t,S_t)\,\mathrm{d}t + f_s(t,S_t)(r(t,S_t)S_t\,\mathrm{d}t + \sigma(t,S_t)S_t\,\mathrm{d}W_t^{\vartheta}) + \frac{1}{2}f_{ss}(t,S_t)\,\mathrm{d}\langle S\rangle_t\right]
\end{aligned}$$

Rearrange to get

$$\begin{aligned}
\mathrm{d}\big(f(t,S_t)B_t^{-1}\big) &= B_t^{-1}\overbrace{\left[f_t(t,S_t) + f_s(t,S_t)r(t,S_t)S_t + \frac{1}{2}f_{ss}(t,S_t)\sigma(t,S_t)^2S_t^2 - r(t,S_t)f(t,S_t)B_t^{-1}\right]}^{=0}\mathrm{d}t \\
&\quad + B_t^{-1}f_s(t,S_t)\sigma(t,S_t)S_t\,\mathrm{d}W_t^{\vartheta}
\end{aligned}$$

On the other hand, we have shown that the discounted wealth has process below, and with the choice of π, we have

$$\begin{aligned}
\mathrm{d}\left(B_t^{-1}X_t\right) &= B_t^{-1}\sigma(t,S_t)\pi_t\,\mathrm{d}W_t^{\vartheta} \\
&= B_t^{-1}\sigma(t,S_t)f_s(t,S_t)S_t\,\mathrm{d}W_t^{\vartheta}
\end{aligned}$$

With initial wealth $f(0,S_0)$, this replicates the price process. Thus $f(t,S_t) = X_t$, and in particular, $X_T = F(S_T)$. □

The Black-Scholes no-arbitrage PDE, together with the terminal condition defines what we call a Cauchy problem, whose solution gives the price of European contingent claims. Solving this Cauchy problem (which we will do next using the Feynman-Kac theorem) yields the celebrated Black-Scholes formula.

4.4 From PDEs to SDEs

We have shown how to go backwards from an expectation to a PDE. In light of the previous theorem, we show the required conditions to obtain a probabilistic representation of the solution to the Cauchy problem (often referred to as the Feynman-Kac representation).

Theorem 4.4.1: Feynman-Kac

Let $X(x,t)$ be a solution to the following SDE

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t$$

with initial condition $X_t = x$. If

- μ, σ are Lipschitz and satisfy the growth condition,
- r is bounded
- f satisfies polynomial growth condition and solves the PDE

$$f_s(s,x) + f_x(s,x)\mu(s,x)\,\mathrm{d}s + \frac{1}{2}f_{xx}(s,x)\sigma(s,x)^2 - r_s(s,x)f(s,x) = 0$$

then

$$f(t, X_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u,X_u)\,\mathrm{d}u}F(X_T)\;|\mathcal{F}_t\right]$$

Proof. Let f be the solution to the Cauchy problem, and X be the solution to $\mathrm{d}X_u = \mu(u, X_u)\,\mathrm{d}u + \sigma(t, X_u)\,\mathrm{d}W_u$ with initial condition $X_t = x$. Given the process X,

- Let

$$R_s = \int_t^s r(u, X_u)\,\mathrm{d}u$$

  and $\mathrm{d}R_s = r(s, X_s)\,\mathrm{d}s$.

- Let $H_s = e^{-R_s}$ be the negative exponential of the short rate, with

$$\mathrm{d}H_s = -e^{-R_s}\,\mathrm{d}R_s = -r_s e^{-R_s}\,\mathrm{d}s$$

- By Itô's lemma,

$$\mathrm{d}Z_s = \left[f_s + f_x\mu + \frac{1}{2}f_{xx}\sigma^2\right]\mathrm{d}s + f_x\sigma\,\mathrm{d}W_s$$

- Let $Y_s = H_s f = e^{-\int_t^s r(u,X_u)\,\mathrm{d}u}f$, thus

$$\begin{aligned}
\mathrm{d}Y_s &= f\,\mathrm{d}H_s + H_s\,\mathrm{d}f + 0\\
&= -r_s f e^{-R_s}\,\mathrm{d}s + e^{-R_s}\left[f_s + f_x\mu\,\mathrm{d}s + \frac{1}{2}f_{xx}\sigma^2\right]\mathrm{d}s + e^{-Rs}f_x\sigma\,\mathrm{d}W_s\\
&= e^{-R_s}\underbrace{\left[-r_s f + f_s + f_x\mu\,\mathrm{d}s + \frac{1}{2}f_{xx}\sigma^2\right]}_{=0}\mathrm{d}s + e^{-R_s}f_x\sigma\,\mathrm{d}W_s
\end{aligned}$$

  Since f satisfies the PDE, we have

$$\mathrm{d}Y_s = e^{-R_S}f_x\sigma\,\mathrm{d}W_s$$

  The polynomial growth condition on f and conditions on σ show that the RHS is a martingale (beyond). Thus given Y also has zero drift, it is a martingale. Hence $Y_t = \mathbb{E}\left[Y_T|\mathcal{F}_t\right]$ (the t at the start).

Note (i) $Y_t = H_t f(t, X_t) = 1 \cdot f(t, X_t)$, and (ii) $Y_T = H_T F(X_T)$ using the PDE terminal conditions. Combining (i) and (ii) we can rewrite $Y_t = \mathbb{E}\left[Y_T | \mathcal{F}_t\right]$ as

$$f(t, X_t) = \mathbb{E}\left[e^{-\int_t^T r(u, X_u)\,\mathrm{d}u} F(X_T)\;|\mathcal{F}_t\right]$$

which is the desired result.

□

4.5 Exercises

4.6 Solutions

Ex.1 (i) Under EMM, the portfolio process evolves under

$$\mathrm{d}X_t = r(t,S_t)X_t\,\mathrm{d}t + \sigma(t,S_t)\pi_t\,\mathrm{d}W_t^{\vartheta}$$

Integrating by parts

$$\begin{aligned}
\mathrm{d}\left(e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}X_t\right) &= X_t\,\mathrm{d}\Big(e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\Big) + e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\,\mathrm{d}X_t + 0 \\
&= -X_t r(t,S_t)\Big(e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\Big)\,\mathrm{d}t \\
&\quad + e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\left[r(t,S_t)X_t\,\mathrm{d}t + \sigma(t,S_t)\pi_t\,\mathrm{d}W_t^{\vartheta}\right] \\
&= e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}\sigma(t,S_t)\pi_t\,\mathrm{d}W_t^{\vartheta}
\end{aligned}$$

Thus we have that $\left(e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}X_t\right)$ is an $(\mathcal{F}_t)$-local martingale under $\mathbb{Q}$ (necessary condition for Itô integrability)

(ii) A portfolio of $\pi_t = S_t$ is equal (a.s.)

Ex.2 Assume $\exists\ \pi$, i.e. an admissible portfolio process π with initial endowment $X_0 = 0$, s.t. the associated self-financing portfolio wealth X satisfying an arbitrage

- for some $T > 0$, $\mathbb{P}\left(X_T \geq 0\right) = 1$
- for some $T > 0$, $\mathbb{P}\left(X_T \geq 0\right) > 0$

As the risk-neutral measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$, thus

- for some $T > 0$, $\mathbb{Q}\left(X_T \geq 0\right) = 1$
- for some $T > 0$, $\mathbb{Q}\left(X_T \geq 0\right) > 0$

We proceed now with the contradiction:

Step 1. **(Conditions imposed by admissibility)** Since the portfolio strategy π is admissible, there exists a constant $\tilde{K} \geq 0$ s.t. $X_T \geq -\tilde{K}$. Since r is bounded from below, there exists a constant $R > 0$ s.t. $r(t,s) \geq -R$ for all $t, s > 0$, thus $e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} \leq e^{Rt} < \infty$.

Combining these two, for all $t \in [0,T]$ we have

$$X_t e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} \geq -\tilde{K}e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} \geq -\tilde{K}e^{Rt} =: -K > -\infty$$

Thus $X_t e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} \geq -K \implies X_t e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} + K \geq 0$ is a positive $(\mathcal{F}_t)$-local martingale with respect to the probability measure $\mathbb{Q}$, and is therefore a **supermartingale**, that is

$$\mathbb{E}^{\mathbb{Q}}\left[X_t e^{-\int_0^t r(u,S_u)\,\mathrm{d}u} + K|\mathcal{F}_0\right] \leq X_0 + K = K$$

In other words,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(u,S_u)\,\mathrm{d}u}X_T\right] &= \left(\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(u,S_u)\,\mathrm{d}u}X_T + K\right] - K\right) \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r(u,S_u)\,\mathrm{d}u}X_T + K|\mathcal{F}_0\right]\right] - K \\
&\leq \mathbb{E}^{\mathbb{Q}}\left[X_0 + K\right] - K \\
&= 0
\end{aligned}$$

Step 2. **(Conditions imposed by arbitrage existence)** As $\int_0^T |r(u,S_u)|\,\mathrm{d}u < \infty$, we have

$$X_T > 0 \iff e^{-\int_0^T r(u,S_u)\,\mathrm{d}u} X_T > 0$$

Since we assume an arbitrage exists under $\mathbb{Q}$, we have

$$\mathbb{Q}\left(e^{-\int_0^T r(u,S_u)\,\mathrm{d}u} X_T \geq 0\right) = 1 \quad \text{and} \quad \mathbb{Q}\left(e^{-\int_0^T r(u,S_u)\,\mathrm{d}u} X_T > 0\right) = 1$$

which implies

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(u,S_u)\,\mathrm{d}u} X_T\right] > 0$$

Since admissibility and arbitrage independently produce contradicting results, the two cannot both be true.

Ex.3 If we can find a replicating portfolio π_t s.t. the associated wealth process

$$X_T(X_0, \pi) = H_T$$

then the value of the portfolio is the same as the asset H_t at time $t \leq T$, that is

$$B_t^{-1} X_t^{\pi} = \frac{1}{L_t} \mathbb{E}^{\mathbb{P}}\left[B_T^{-1} L_T H_T | \mathcal{F}_t\right]$$

where $B_t := e^{\int_0^t r(u,S_u)\,\mathrm{d}u}$. Now consider the $(\mathcal{F}_t, \mathbb{P})$-martingale M defined by

$$M_t := \mathbb{E}^{\mathbb{P}}\left[B_T^{-1} L_T H_T | \mathcal{F}_t\right]$$

s.t. $B_t^{-1} X_t = \frac{M_t}{L_t}$. Then by the martingale representation theorem, there exists a K that is $(\mathcal{F}_t)$-adapted s.t. for all $t \geq 0$

$$M_t = \mathbb{E}^{\mathbb{P}}\left[B_T^{-1} L_T H_T\right] + \int_0^t K_s \,\mathrm{d}W_s \iff \mathrm{d}M_t = K_t \,\mathrm{d}W_t = K_t(\mathrm{d}W_t^{\vartheta} - \theta\,\mathrm{d}t)$$

Now recall that $\mathrm{d}L_t = -\theta L_t \,\mathrm{d}W_t$, which by Itô's lemma implies

$$\mathrm{d}L_t^{-1} = \theta^2 L_t^{-1}\,\mathrm{d}t + \theta L_t^{-1}\,\mathrm{d}W_t = \theta L_t^{-1}\,\mathrm{d}W_t^{\vartheta}$$

Integrating by parts, we have

$$\begin{aligned}
\mathrm{d}\left(\frac{M_t}{L_t}\right) &= M_t\,\mathrm{d}L_t^{-1} + L_t^{-1}\,\mathrm{d}M_t + \theta K_t L_t^{-1}\,\mathrm{d}t \\
&= M_t \theta L_t^{-1}\,\mathrm{d}W_t^{\vartheta} + L_t^{-1} K_t\,\mathrm{d}W_t^{\vartheta} - L_t^{-1}\theta K_t\,\mathrm{d}t + \theta K_t L_t^{-1}\,\mathrm{d}t \\
&= (M_t\theta + K_t) L_t^{-1}\,\mathrm{d}W_t^{\vartheta}
\end{aligned}$$

and

$$\frac{M_t}{L_t} = \frac{M_0}{L_0} + \int_0^t \frac{M_s\theta + K_s}{L_s}\,\mathrm{d}W_s^{\vartheta}$$

Consider the self-financing strategy π_t with initial wealth

$$X_0 = M_0 \qquad \text{(which is also just } \frac{M_0}{L_0}\text{)}$$

and process defined by

$$\pi_t := B_t \left(\frac{M_t \theta + K_t}{\sigma(t, S_t) L_t} \right)$$

Therefore using the portfolio evolution definition, the discounted wealth process can be written as

$$\begin{aligned} B_t^{-1} X_t &= X_0 + \int_0^t B_t^{-1} \pi_u \sigma(u, S_u) \, \mathrm{d}W_u^{\vartheta} \\ &= \frac{M_0}{L_0} + \int_0^t \frac{M_s \theta + K_s}{L_s} \, \mathrm{d}W_s^{\vartheta} \end{aligned}$$

We showed above that the RHS can be expressed simply as M_t / L_t through repeated application of Itô calculus. Therefore at time T, we have that X_T by our definition of M_t is

$$\begin{aligned} X_T &= B_T \frac{M_T}{L_T} \\ &= \frac{B_T}{L_T} \mathbb{E}^{\mathbb{Q}} \left[B_T^{-1} L_T H_T | \mathcal{F}_T \right] \\ &= H_T \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Therefore since there exists a replicating portfolio, then the price at time t, H_t must be equal to the wealth process at time t, X_t.

Chapter 5

Volatility and Calibration

5.1 Historical Volatility

5.1.1 Spot volatility

In our generalised Black-Scholes model, the stock process under the risk-neutral measure is

$$\mathrm{d}S_t = r(t, S_t)S_t\,\mathrm{d}t + \sigma(t, S_t)S_t\,\mathrm{d}W_t^{\vartheta}$$

As the risk-free rate can be read off the market, the only parameter we need to estimate is σ_t, the (spot) volatility.

Definition 5.1.1: Spot volatility

Spot volatility, σ can be defined as

$$\sigma(t, S_t)^2 = \frac{\mathrm{d}\langle \log S\rangle_t}{\mathrm{d}t}$$

in models **without jumps**.

Proof. By Itô's lemma,

$$\begin{aligned}\mathrm{d}\log S_t &= \frac{1}{S_t}\,\mathrm{d}S_t - \frac{1}{2}\frac{1}{S_t^2}\,\mathrm{d}\langle S\rangle_t \\ &= \left(\mu(t, S_t) - \frac{\sigma(t, S_t)^2}{2}\right)\mathrm{d}t + \sigma(t, S_t)\,\mathrm{d}W_t^{\vartheta}\end{aligned}$$

Quadratic variation is therefore

$$\mathrm{d}\langle \log S\rangle_t = \sigma_t^2\,\mathrm{d}t$$

□

5.1.2 Constant parameter estimation

We discretise our model; denote counter $i = 1, \ldots, n$ and interval Δ s.t. $n\Delta = T$. Define log-returns of an interval:

$$r_i = \log\left(\frac{S_i}{S_{i-1}}\right)$$

Lemma 5.1.2: Return distribution

Log returns are distributed normally with

$$r_i \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta, \sigma^2\Delta\right)$$

assuming μ, σ are constant on the interval $t \in [0, T]$

Proof for Lemma

We have

$$\begin{aligned} r_i &= \log(S_i) - \log(S_{i-1}) \\ &= \int_{\Delta(i-1)}^{\Delta i} \left(\left(\mu - \frac{\sigma^2}{2}\right) \mathrm{d}t + \sigma \, \mathrm{d}W_t^{\vartheta}\right) \end{aligned}$$

The result follows trivially from the properties of the Itô integral. ■

Theorem 5.1.3

The maximum likelihood estimator, $\hat{\sigma}^2$ of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n\Delta} \sum_{i=1}^{n} (r_i - \bar{r})^2$$

where $\bar{r} := \frac{1}{n} \sum_{i=1}^{n} r_i$.

Proof. tbc. □

5.1.3 General parameter estimation

In general μ, σ are not constant. In this case we can use the fact quadratic variation is finite to estimate the integral of variance over 0 to T.

Theorem 5.1.4

We have

$$\sum_{i=1}^{n} r_i^2 \overset{\mathcal{L}^2}{\to} \int_0^T \sigma(t, S_t)^2 \, \mathrm{d}t$$

that is for $X_n := \sum_{i=1}^{n} r_i^2$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left|X_n - \int_0^T \sigma(t, S_t)^2 \, \mathrm{d}t\right|^2\right] = 0$$

Proof. From the proof above, in integral form we have

$$\begin{aligned} \int_0^T \sigma(t, S_t) \, \mathrm{d}t &= \langle S \rangle_T \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\log(S_{i\Delta}) - \log(S_{(i-1)\Delta})\right]^2 \end{aligned}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} r_i^2$$

□

Convergence in $\mathcal{L}^2$ implies convergence in probability, thus we can estimate $\int_0^T \sigma(t, S_t)^2 \,\mathrm{d}t$ consistently by $\sum_{i=1}^n r_i^2$ with a large number of observations on interval $[0, T]$. This estimator is better as it gives us convergence a.s. in the limit.

Note however that our model of stock prices based on Brownian motion **is not a good model of tick level/super high-frequency data** due to large jumps, i.e. fractals but also other issues such as noise, autocorrelation, microstructure noise...

5.2 Implied volatility

Definition 5.2.1: Implied volatility surface

The *implied volatility*, $\Sigma_t(T, K)$ is the unique volatility that solves

$$C^{BS}(\Sigma_t^2; t, S_t, T, K) = C_t^{\text{mkt}}(T, K)$$

We call the function Σ_t the (implied) *volatility surface*.

Implied volatility is a measure of current/future volatility, and as such, it is forward looking; as opposed to historical/realised volatility which measure known past realisations.

Remark.

> Under the vanilla Black-Scholes assumptions, the implied volatility surface should be constant.

5.2.1 VIX

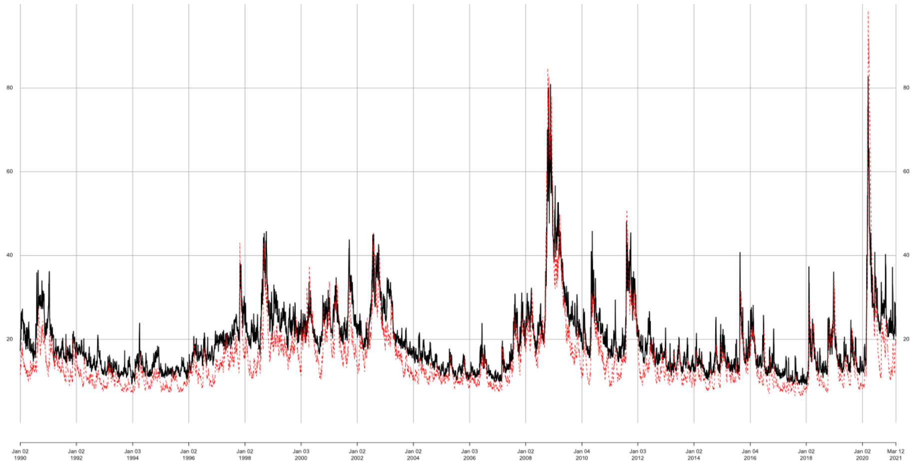

Figure 5.1: VIX (black) vs. historical (red)

According to the CBOE (Chicago Board Options Exchange):

> "The VIX Index is a financial benchmark designed to be an up-to-the-minute market estimate of expected volatility of the S&P 500 Index, and is calculated by using the midpoint of real-time S&P 500 Index

> (SPX) option bid/ask quotes. More specifically, the VIX Index is intended to provide an instantaneous measure of how much the market thinks the S&P 500 Index will fluctuate in the 30 days from the time of each tick of the VIX Index."

Thus for a single stock, the VIX estimates

$$\frac{1}{\Delta}\mathbb{E}^{\mathbb{Q}}\left[\int_t^{t+\Delta}\sigma_t^2\,\mathrm{d}t|\mathcal{F}_t\right]$$

where Δ corresponds to 30 days in a year, and $\mathbb{Q}$ is the implied martingale measure.

Practically, the VIX is a weighted average of the implied volatilities of a range of SPX liquidly traded options (this is why it is sometimes referred to as 30 days implied volatility).

5.3 Evidence on Black-Scholes

5.3.1 Stylised facts

Some empirical observations on stock/option prices:

- Volatility clustering and persistence: small price moves follow small moves, large moves follow large moves (high autocorrelation).
- Thick tails: distribution of asset returns have heavier tails than normal distribution (leptokurtic distribution).
- Negative correlation between prices and volatility: when prices go down, volatility tends to rise (leverage effect).
- Mean reversion: volatility tends to revert to some long-run level.

The Black-Scholes model obviously does not capture these stylized facts.

5.3.2 Volatility Smile/Smirk

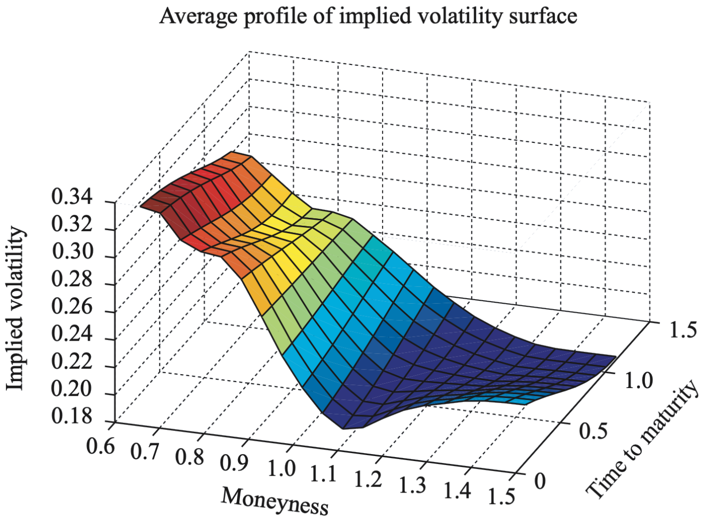

Figure 5.2: Implied volatility surface of S&P500 options, as a function of K and T in March 1999; (Cont and da Fonseca, 2002)

The BS assumption of constant implied volatility clearly does not hold in markets where calls and puts are liquidly traded (otherwise, implied volatilities cannot be observed). Figure 5.2 shows that implied volatilities today change as maturity and strike changes. Moreover, implied volatilities for fixed maturities and strikes also change over time; that is, $\Sigma_t(T, K)$ as a function of t is not constant.

5.4 Local volatility calibration

We use a simplified version of the local volatility/generalised Black-Scholes model developed in Chapter 4.

Assume the price of the stock follows

$$\frac{\mathrm{d}S_t}{S_t} = \mu(t, S_t)\,\mathrm{d}t + \sigma(t, S_t)\,\mathrm{d}W_t$$

and the money market follows

$$\mathrm{d}B_t = rB_t\,\mathrm{d}t$$

In addition:

- μ and σ are **deterministic functions** s.t. there is a unique strong solution to the stock price process, and the solution is Markov.
- σ is bounded away from zero
- r is **constant**
- the process L defined as

  $$L_t := \exp\left(-\int_0^t \frac{\mu(u, S_u) - r}{\sigma(u, S_u)}\,\mathrm{d}W_t - \frac{1}{2}\int_0^t \left(\frac{\mu(u, S_u) - r}{\sigma(u, S_u)}\right)^2 \mathrm{d}u\right)$$

  is **well-defined** and a **martingale**.

5.4.1 Calibration

Definition 5.4.1: Calibration

Calibration is the process of finding a local volatility function, $\sigma(t, s)$ s.t. the model prices agree with the observed market prices.

Suppose we have liquidly traded calls for all strikes and maturities, i.e. we have time t market prices for

$$C_t^{\mathsf{mkt}}(u, k)$$

for $k \in \mathbb{R}_+$ and all $u \in [t, T]$. We want to find $\sigma : (s, x) \to \sigma(s, x)$ s.t.

$$C^{LV}(\{\sigma(s, x)\}_{s\in[t,u], x\in[0,+\infty)}, t, S_t, u, k) = C^{\mathsf{mkt}}(t, u, k)$$

for all $k \in \mathbb{R}_+$ an all $u \in [t, T]$.

Theorem 5.4.2: Dupire formula

The local volatility the matches the given market call price is

$$\sigma_t(u,k)^2 = 2\frac{\partial_u C_t^{\mathsf{mkt}}(u,k) + rk\partial_k C_t^{\mathsf{mkt}}(u,k)}{k^2\partial_{kk}C_t^{\mathsf{mkt}}(u,k)}$$

Proof. We break the proof down into steps:

Step 1. Note the stock evolution has a unique strong solution and S is Markov. Thus

$$\begin{aligned}\mathbb{Q}\left(S_u \le x\right) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{S_u\le x\}}|\mathcal{F}_t\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{S_u\le x\}}|S_t\right] \quad \text{(Markov)}\\ &= F^{\mathbb{Q}}(x,u;y,t)\end{aligned}$$

where $F^{\mathbb{Q}}(x,u;y,t)$ is the CDF of $S_t \le x$ under measure $\mathbb{Q}$ if we started S at y at time t. The derivative w.r.t. x gives us the PDF of S_u under $\mathbb{Q}$; define this as $f(x,u;y,t)$. Note

$$f(0,u;y,t) = f(+\infty,u;y,t) = 0$$

that is probability the price at u is 0 or ∞ is zero.

Using the formula for call prices

$$\begin{aligned}C_t^{\mathsf{mkt}}(u,k) &= C^{\mathsf{LV}}\\ &= e^{-r(u-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_u-k)^+|\mathcal{F}_t\right]\\ &= e^{-r(u-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_u-k)^+|S_t\right]\\ &= e^{-r(u-t)}\int_k^\infty (x-k)f(u,x;t,S_t)\,\mathrm{d}x\end{aligned}$$

Differentiate w.r.t. k to get

$$\partial_k C_t^{\mathsf{mkt}}(u,k) = -e^{-r(u-t)}\int_k^\infty f(u,x;t,S_t)\,\mathrm{d}x \tag{5.4.1}$$

$$\partial_{kk} C_t^{\mathsf{mkt}}(u,k) = e^{-r(u-t)}f(u,k;t,S_t) \tag{5.4.2}$$

Therefore from observing call prices, we can deduce the distribution of S_u given S_t, and we can price all European claims since we know the marginal distributions under the equivalent martingale measure.

Step 2. Next consider any bounded function h and

$$v(u,S_u) = \mathbb{E}^{\mathbb{Q}}\left[h(S_T)|S_u\right] = \mathbb{E}^{\mathbb{Q}}\left[h(S_T)|\mathcal{F}_u\right]$$

By the Feynman-Kac theorem (note no discount):

$$v_u(u,x) + rxv_x(u,x) + \frac{1}{2}\sigma^2(u,x)x^2v_{xx}(u,x) = 0$$

The expectation is also zero,

$$\int_0^\infty \left[v_u(u,x) + rxv_x(u,x) + \frac{1}{2}\sigma^2(u,x)x^2v_{xx}(u,x)\right]f(u,x;t,S_t)\,\mathrm{d}x = 0$$

Term by term, we integrate by parts;

- For the first term, note

  $$v(t,S_t)=\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[h(S_T)|S_u\right]|S_t\right]=\int_0^\infty v(u,x)f(u,x;t,S_t)\,\mathrm{d}x$$

  therefore the derivative w.r.t. to u is

  $$\begin{aligned}0&=\int_0^\infty \partial_u v(u,x)f(u,x;t,S_t)\,\mathrm{d}x+\int_0^\infty v(u,x)\partial_u f(u,x;t,S_t)\,\mathrm{d}x\\ \implies \int_0^\infty \partial_u v(u,x)f(u,x;t,S_t)\,\mathrm{d}x&=-\int_0^\infty v(u,x)\partial_u f(u,x;t,S_t)\,\mathrm{d}x\end{aligned}$$

- For the second term,

  $$\begin{aligned}\int_0^\infty rxv_x(u,x)f(u,x;t,S_t)\,\mathrm{d}x&=\int_0^\infty rxf(u,x;t,S_t)\,\mathrm{d}v(u,x)\\ &=\underbrace{[v(u,x)rxf(u,x;t,S_t)]_0^\infty}_{=0}-\int_0^\infty r\partial_x\{xf(u,x;t,S_t)\}v(u,x)\,\mathrm{d}x\\ &=-\int_0^\infty r\partial_x\{xf(u,x;t,S_t)\}v(u,x)\,\mathrm{d}x\end{aligned}$$

  (assuming we can pass the derivative inside).

- For the second derivative term, we integrate twice

  $$\begin{aligned}&\int_0^\infty \frac{1}{2}\sigma^2(u,x)x^2v_{xx}(u,x)f(u,x;t,S_t)\,\mathrm{d}x\\ &\quad=\left[\frac{1}{2}\sigma^2(u,x)x^2v_x(u,x)f(u,x;t,S_t)\right]_0^\infty-\int_0^\infty v_x(u,x)\partial_x\{\frac{1}{2}\sigma^2(u,x)x^2f(u,x;t,S_t)\}\,\mathrm{d}x\end{aligned}$$

  The first term on the RHS is zero, integrating the second term by parts again,

  $$=-\left[v(u,x)\partial_x\{\frac{1}{2}\sigma^2(u,x)x^2f(u,x;t,S_t)\}\right]_0^\infty+\int_0^\infty v(u,x)\partial_{xx}\{\frac{1}{2}\sigma^2(u,x)x^2f(u,x;t,S_t)\}\,\mathrm{d}x$$

  Assuming again the first term converges to zero, we are left the integral.

Combining the three parts back, we have

$$\int_0^\infty v(u,x)\left[-\partial_u f(u,x;t,S_t)-\partial_x f(u,x;t,S_t)+\frac{1}{2}\partial_{xx}\{\sigma^2(u,x)x^2f(u,x;t,S_t)\}\right]\mathrm{d}x=0$$

As h was arbitrary, we obtain the backward Kolmogorov equation for transition probability

$$\partial_u f(u,x;t,S_t)+r\partial_x\{xf(u,x;t,S_t)\}-\frac{1}{2}\partial_{xx}\{\sigma^2(u,x)x^2f(u,x;t,S_t)\}=0 \tag{5.4.3}$$

Step 3. On the other hand, differentiate $C_t^{\mathsf{mkt}}(u,k)$ w.r.t. u and substitute (5.4.3) above into the second integral to get

$$\begin{aligned}\partial_u C_t^{\mathsf{mkt}}(u,k)&=-r\underbrace{e^{-r(u-t)}\int_k^\infty (x-k)f(u,x;t,S_t)\,\mathrm{d}x}_{C_t^{\mathsf{mkt}}(u,k)}+e^{-r(u-t)}\int_k^\infty (x-k)\partial_u f(u,x;t,S_t)\,\mathrm{d}x\\ &=-rC_t^{\mathsf{mkt}}(u,k)+e^{-r(u-t)}\int_k^\infty (x-k)\left[-\partial_x\{xf(u,x;t,S_t)\}+\frac{1}{2}\partial_{xx}\{\sigma^2(u,x)x^2f(u,x;t,S_t)\}\right]\mathrm{d}x\end{aligned}$$

$$\begin{aligned}&= -rC_t^{\mathsf{mkt}}(u,k) - e^{-r(u-t)} \overbrace{\int_k^\infty (x-k) r\partial_x\{xf(u,x;t,S_t)\}\,\mathrm{d}x}^{(i)} \\ &\quad + \underbrace{e^{-r(u-t)} \int_k^\infty (x-k)\frac{1}{2}\partial_{xx}\{\sigma^2(u,x)x^2 f(u,x;t,S_t)\}\,\mathrm{d}x}_{(ii)}\end{aligned}$$

(i) can be written as

$$\int_k^\infty (x-k) r\partial_x\{xf(u,x;t,S_t)\}\,\mathrm{d}x = \big[(x-k)xf(u,x;t,S_t)\big]_0^\infty - \int_0^\infty xf(u,x;t,S_t)\,\mathrm{d}x$$

(ii) can be written as

$$\begin{aligned}&\int_k^\infty (x-k)\frac{1}{2}\partial_{xx}\{\sigma^2(u,x)x^2 f(u,x;t,S_t)\}\,\mathrm{d}x \\ &\quad = \underbrace{\left[(x-k)\frac{1}{2}\partial_x\{\sigma^2(u,x)x^2 f(u,x;t,S_t)\}\right]_0^\infty}_{=0} - \int_0^\infty \frac{1}{2}\partial_x\{\sigma^2(u,x)x^2 f(u,x;t,S_t)\}\,\mathrm{d}x \\ &\quad = \frac{1}{2}\sigma^2(u,k)k^2 f(u,k;t,S_t)\end{aligned}$$

Therefore we have

$$\partial_u C_t^{\mathsf{mkt}}(u,k) = -rC_t^{\mathsf{mkt}}(u,k) + e^{-r(u-t)}\int_0^\infty rxf(u,x;t,S_t)\,\mathrm{d}x + e^{-r(u-t)}\frac{1}{2}\sigma^2(u,k)k^2 f(u,k;t,S_t)$$

Substitute definitions (5.4.1) and (5.4.2) for market call derivatives

$$\implies \partial_u C_t^{\mathsf{mkt}}(u,k) = -rC_t^{\mathsf{mkt}}(u,k) + e^{-r(u-t)}\partial_k\{C_t^{\mathsf{mkt}}(u,k)\} + e^{-r(u-t)}\frac{1}{2}\sigma^2(u,k)k^2\partial_{kk}\{C_t^{\mathsf{mkt}}(u,k)\}$$

Rearrange to obtain Dupire's formula

$$\sigma_t(u,k)^2 = 2\frac{\partial_u C_t^{\mathsf{mkt}}(u,k) + rk\partial_k C_t^{\mathsf{mkt}}(u,k)}{k^2\partial_{kk}C_t^{\mathsf{mkt}}(u,k)}$$

□

Takeaways: (1) Observing call prices provides restrictions on stock price evolution under the risk-neutral measure; (2) the transition density function follows a Kolmogorov PDE, i.e. call prices give us information about the future distribution of prices.

5.4.2 Problems with local volatility

Local volatility models can perfectly fit marginals (European-style path-independent options), but the calibration is very unstable. It has significant problems with pricing path-dependent options, and since dynamics of local volatilities are not realistic and they tend to change drastically with re-calibrations.

5.5 Exercises

5.6 Solutions

Chapter 6

Appendix

Lemma 6.0.1: Supremum over real interval equal to rational interval

Let $f : \mathbb{R} \to \mathbb{R}$ be a right-continuous function. Then

$$\sup_{s \in [a,b]} f(s) = \sup_{s \in [a,b] \cap \mathbb{Q}} f(s)$$

Proof for Lemma

Bibliography

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