FM213 MT Handbook Ver. 1.2

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Contents

Derivations can be found in appendix for subsections with *

1 Present Values and Discounting

1.1 Present Value (PV)

The present value of a **single cashflow** C_t , with **discount rate** r arriving at **time** t, is given by the **present value function** $PV(C_t)$ where:

$$PV(C_t) = \frac{C_t}{(1+r)^t}$$

or expressed in words:

$$\left[\text{Present value of } C_t\right] = \frac{\left[\text{Cashflow at time } t\right]}{\left[1+r\right]^t}$$

The discount rate r represents the investors required return for one period to compensate for the level of risk C_t possesses. It is the (MRS, marginal rate of substitution) of investment across time, or the opportunity cost of investing in a project with a given risk.

1.2 Discount Factor (DF)

We define the **discount factor** DF, or δ as follows:

$$DF, \delta = \frac{1}{(1+r)}$$

and so $PV(C_t) = \delta^t \times C_t$

1.3 Net Present Value (NPV)

If an **investment/project A** has **cashflows** C_t **across time** t, then A has a Net Present Value (NPV) of:

$$NPV(A) = \sum_{t=0}^{T} \frac{C_t}{(1+r)^t}$$

where r = discount rate for the project

NB. C_t can be **negative representing a cost** to the investment/project.

If $NPV(A) > 0 \Rightarrow$ a net profit is made, i.e. this investment will make us money
$NPV(A) < 0 \Rightarrow$ a net loss is made, i.e. this investment will lose us money

Hence only invest if $NPV(A) > 0$

2 Standard PV/Rates Calculations

2.1 Perpetuity (without growth)*

A perpetuity is an agreement to pay a **fixed amount** each period, **starting next period, forever**. Its present value can be represented by:

$$PV(P) = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

where:

C = periodic/(or annual) cashflow/payment

r = discount rate for this project, i.e. a risk-adjusted discount rate

2.2 Perpetuity (with growth)*

Instead of constant payments, let C **grow at a rate of** g **each period**, hence, $C_t = C(1+g)^{t-1}$ and so its present value can be represented by

$$PV(P_g) = \sum_{t=1}^{\infty} \frac{C(1+g)^{t-1}}{(1+r)^t} = \frac{C}{r-g}$$

2.3 Annuities (without growth)*

An annuity is an agreement to pay a **fixed amount each period, starting next period, for a finite number of periods** (rather than infinite like a perpetuity).

Let T be the period of the last payment, hence an annuity can be represented by

$$PV(A) = \sum_{t=1}^{T} \frac{C}{(1+r)^t} = \frac{C}{r}\left[1 - \frac{1}{(1+r)^T}\right]$$

2.4 Annuity (with growth)*

Instead of constant payments, let C **grow at a rate of** g **each period**, hence, $C_t = C(1+g)^{t-1}$ and so its present value can be represented by

$$PV(A_g) = \sum_{t=1}^{T} \frac{C(1+g)^{t-1}}{(1+r)^t} = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right]$$

2.5 Quoted vs Effective rates

Quoted Rate

- The K month interest rate is quoted at r
- The amount received over the K month(s) is $\frac{K}{12} \times r$

eg Skeeter lends Nimar £100, at 6% interest over 1 month.
Nimar will pay back Skeeter: $\frac{1}{12} \times 6\,\% = 0.5\,\%$ interest over the 1 month which amounts to £100.50

Effective Rate

The Effective Annual Rate (EAR) is the **actual interest received on an annual basis**, essentially converting interest rates into the intuitive standard annual rate .

Let r be the K month quoted rate. If we were to compound the interest N times a year, then $N = \frac{12}{K}$

Hence the EAR is:

$$1 + EAR = \left[1 + \frac{r}{N}\right]^N$$

2.6 Continuous Compounding and Discounting*

If we compound more and more frequently, i.e. let $N \to \infty$, then the EAR becomes:

$$\lim_{N\to\infty} [1 + EAR] = \lim_{N\to\infty} \left[1 + \frac{r}{N}\right]^N = e^r$$

Hence:

£100 **compounded continuously** at 5% for 1 year is $100e^{0.05}$

and £100 **compounded continuously** at 5% for 2 years is $100e^{0.05\times2} = 100e^{0.1}$

The future value of an investment today, X, that compounds continuously for t years at quoted rate r is

$$FV_t(X) = Xe^{rt}$$

Reversing the logic, the **present value of a cash flow C_t at time** t is

$$PV(C_t) = \frac{C_t}{e^{rt}} = C_t e^{-rt}$$

NB. Note the minus sign when discounting.

2.7 Real and Nominal Rates (Fisher Relationship)*

Let

π = inflation rate
i = nominal interest rate
r = real interest rate

The real interest rate is given by:

$$[1 + r] = \frac{[1 + i]}{[1 + \pi]}$$

Because interest rates are very small numbers we can approximate with:

$$r \approx i - \pi$$

Discount:
Real cashflows with the real rate
Nominal cashflows with nominal rate

3 Bonds and the Term Structure of Rates

3.1 Introduction and Value of Bond

A bond is a security, sold by governments or corporations that promises fixed payments to investors. It can be thought of as a loan from the market to the issuer.

The security contract stipulates that **coupons payments (fixed payments) are made each period** and **at maturity, the principal is repaid.**

Let

c = coupon rate
F = face value or principal, i.e. amount lent out
T = maturity date (year)
N = frequency of coupons per year

y = yield to maturity (per year)
$P_B = PV_0(B)$ = price of bond (or present value of cashflows from bond)

Then the bond value is the present value of all the future cashflows:

$$P_B = \frac{\frac{cF}{N}}{1 + \frac{y}{N}} + \frac{\frac{cF}{N}}{[1 + \frac{y}{N}]^2} + \dots \frac{\frac{cF}{N} + F}{[1 + \frac{y}{N}]^{NT}}$$

If the bond pays a **coupon annually, i.e.** $N = 1$, the formula reduces to:

$$P_B = \frac{cF}{1 + y} + \frac{cF}{[1 + y]^2} + \dots \frac{cF + F}{[1 + y]^T}$$

3.2 Manual Calculation of Value

The formula above can be decomposed into **the sum of an annuity and the PV of the principal** for easier human calculation using the sum formulae.

The result is:

$$P_B = \frac{cF}{y}\left[1 - \frac{1}{(1 + \frac{y}{N})^{NT}}\right] + \frac{F}{(1 + \frac{y}{N})^{NT}}$$

3.3 Macaulay Duration (D)*

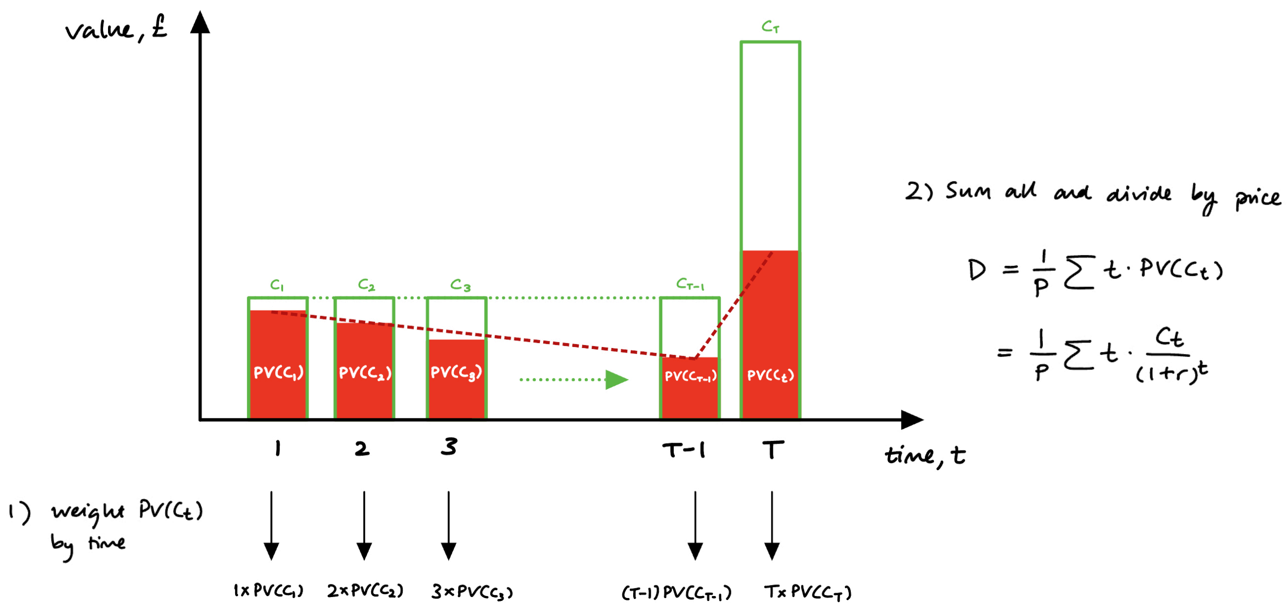
Duration is a measure of the sensitivity of the bond price in response to a change in yield.

Higher Duration = P more sensitive to Δy
Lower Duration = P less sensitive to Δy

Let D = Macaulay Duration, the negative of the elasticity of bond price with respect to yield

$$D = \frac{1}{P}\sum_{t=1}^{T} t \cdot \frac{C_t}{(1+y)^t}$$

Think about it as the **time weighted centre of mass of the** $PV(C_t)$



3.4 Modified Duration (D^M)*

Let D^M **denote Modified Duration**, representing the **percentage change in price** for a given change in yield.

This is important as changes in yields are often quoted in basis points (bps), rather than a percentage change in the rate. 1bp = 0.01%

$$D^M = -\frac{\frac{dP}{dy}}{P} = \frac{D}{1+y}$$

and so by definition:

$$\% \, \Delta P = \frac{\Delta P}{P} \approx -D^M \cdot \Delta y$$

Hence: **Macaulay duration is the elasticity**
Modified duration is the volatility

They are very closely related ideas!

3.5 Spot Rates

Let the annual interest rate to borrow **today** for t years be denoted r_t
r_t **is also known as a SPOT RATE because the loan begins today**

The present value of a cashflow in year t is:

$$PV(C_t) = \frac{C_t}{(1+r_t)^t}$$

Hence we can also define the value of a **bond** with annual coupon rate c and face value F as:

$$P_B = \frac{cF}{1+r_1} + \frac{cF}{(1+r_t)^t} + \dots \frac{cF+F}{(1+r_T)^T}$$

which generalised with cashflow C_t becomes:

$$P_B = \sum_{t=1}^{T} \frac{C_t}{(1+r_t)^t}$$

3.6 Forward Rates

A forward rate is the interest rate f_t , agreed today on a loan starting $(t-1)$ years in the future, repaid the following year in year t.

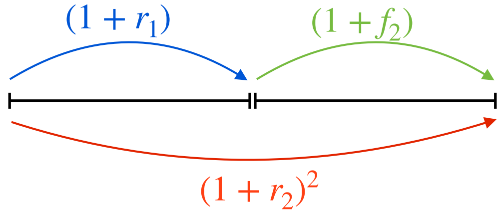
By the Fundamental Theorem of Finance (i.e. no free lunch or no arbitrage exists), we know:

$$(1+r_t)^t = (1+r_{t-1})^{t-1} \cdot (1+f_t)$$

AND

$$(1+r_t)^t = (1+r_1)(1+f_2)(1+f_3)\dots(1+f_t)$$

Graphical example over the first two years:



3.7 How to extract rates from prices

Spot Rate
If given at least two coupon bonds with same maturity, we can calculate the spot rate.

The spot rate is the discount rate for a zero coupon bond (a bond that only pays the principal at maturity)

e.g.

Two bonds, A and B, maturity T

Bond A :

P_A = Price of A
y_A = yield on A
c_A = coupon rate of A
F_A = face value or principal of A

Bond B:

P_B = Price of B
y_B = yield on B
c_B = coupon rate of B
F_B = face value or principal of B

Hence:

$$P_A = \frac{c_A F_A}{1 + y_A} + \frac{c_A F_A}{(1 + y_A)^2} + \ldots \frac{c_A F_A + F_A}{(1 + y_A)^T}$$

$$P_B = \frac{c_B F_B}{1 + y_B} + \frac{c_B F_B}{(1 + y_B)^2} + \ldots \frac{c_B F_B + F_B}{(1 + y_B)^T}$$

Main idea: Create a zero coupon bond by scaling one bond then subtracting the other bond.
i.e. we want to multiply $c_A F_A$ by k to get $k \times c_A F_A = c_B F_B$ which makes the coupons equal.

Hence our cashflows are

Bond	Price	Cashflow in year 1	Cashflow in year <T	Cashflow in year T
A	P_A	$c_A F_A$	$c_A F_A$	$c_A F_A + F_A$
B	P_B	$c_B F_B$	$c_B F_B$	$c_B F_B + F_B$
kA -B	$kP_A - P_B$	$kc_A F_A - c_B F_B = 0$	0	$kF_A - F_B$

The spot rate, r_T, maps the cashflow $kF_A - F_B$ to the price $kP_A - P_B$ by PV relation:

$$[kP_A - P_B] = \frac{kF_A - F_B}{[1 + r_T]^T}$$

where

k makes the coupons equal, so $k \times c_A F_A = c_B F_B$

Forward Rate
Two bonds, A and B, **maturity 2**

Bond A :

P_A = Price of A
c_A = coupon rate of A
F_A = face value or principal of A

Bond B:

P_B = Price of B
c_B = coupon rate of B
F_B = face value or principal of B

Discount Rates (to be found):

r_1, r_2, f_2

We know:

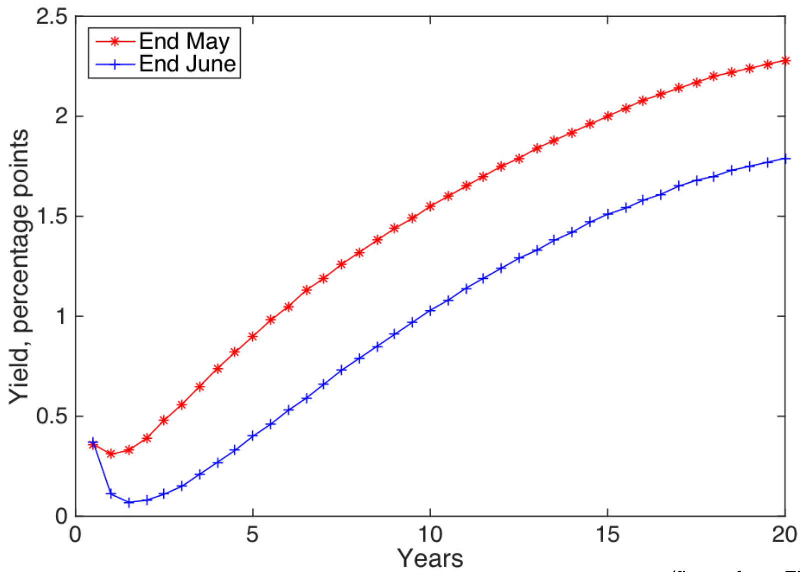
$$P_A = \frac{c_A F_A}{1+r_1} + \frac{c_A F_A + F_A}{(1+r_1)(1+f_2)}$$

$$P_B = \frac{c_B F_B}{1+r_1} + \frac{c_B F_B + F_B}{(1+r_1)(1+f_2)}$$

This is just a set of simultaneous equations that can be solved trivially.

3.8 Term Structure

The longer dated the spot rate, the higher the interest rate.



(figure from FM213 slides, Peng, 2022)

Two reasons together explain why:

Expectations Theory: We expect spot rates to be higher in the future.
Spot rates are the product of forward rates. If we expect short term spot rates to rise in the future, then the forward rate will be higher, which raises the long term spot rate

Liquidity/Term Premium Theory: The longer the maturity, the higher the chance of default meaning there is a preference for short term debt over long term debt. Hence we charge higher interest rates for longer dated spot rates.

3.9 Useful/Key relationships

Bond Price and Face Value
When:

coupon yield, c $=$ yield to maturity, y

$\Rightarrow$ [BOND PRICE, P_A] = [FACE VALUE, F_B]

Bond Price and Yield
When:

[YIELDS $\uparrow$] $\Rightarrow$ [PRICE $\downarrow$]

[YIELDS $\downarrow$] $\Rightarrow$ [PRICE $\uparrow$]

4 Stocks

4.1 Introduction and Definitions

A Stock/Common Stock is a security representing a share of ownership in a company

Let

P_t = price of a stock
N = number of shares in circulation

Then we define:

Dividend at time t, D_t	Payments made by the company to the shareholder, paid out of company earnings	D_t
Dividend yield	Dividend as percentage of stock price	$d_t = \frac{D_t}{P_t}$
EPS (Earnings per share)	Profit made per share	$EPS = \frac{E_t}{N}$ where E_t is the firms total earnings
Market Value	Total value of the firm according to the stock market	Market Capitalisation/Value $= P_t N$
Book Value	Accounting value of the firm	Total Assets - Total Debts (or Liabilities)
P/E ratio	How many multiples of earnings is the price currently	P/E ratio $= \frac{P_t}{EPS}$
Payout ratio	How much of earnings is paid in dividends to shareholders	Payout ratio $= \frac{D_t}{E_t}$
Plowback ratio, ρ	Percentage of earnings, reinvested in firm	ρ = 1- [payout ratio] = $1 - \frac{D_t}{E_t}$
Return on Equity (ROE)	Amount of earnings, £1 of equity produces, i.e. percentage output of firm.	$ROE = \frac{E_t}{\text{Book Value}}$
g, growth rate of earnings	rate at which firms earnings is increasing,	$g = ROE \times$ Plowback ratio

4.2 Basic Valuation Theory*

The price of a stock today, P_S is the **sum of the present values of the expected dividends in the future,** which we can write as:

$$P_S = \sum_{t=1}^{\infty} \frac{\mathbb{E}[D_t]}{[1 + r_S]^t}$$

where:

$P_S =$ price of stock today

$\mathbb{E}[D_t] =$ expected dividend at time t

$r_S =$ required rate of return for stock (risk-adjusted discount rate)

4.3 Constant Dividend Model

Carrying on from 4.2, if we **assume constant guaranteed dividends** of D, then the price is a PV of a perpetuity with cashflow D, and discount rate r_S :

$$P_S = \sum_{t=1}^{\infty} \frac{D}{[1 + r_S]^t} = \frac{D}{r_S}$$

4.4 Growing Dividend Model (aka Gordon Growth Model)

Instead, if we let D_t **grow at a rate** g**, then** $D_t = D_1(1 + g)^{t-1}$ **where** D_1 **is the next/first dividend received.**

Hence we use the **perpetuity with growth** formula:

$$P_S = \sum_{t=1}^{\infty} \frac{D_1[1 + g]^{t-1}}{[1 + r_S]^t} = \frac{D_1}{r_S - g}$$

4.5 Growth rate of Earnings/Dividends/Book Value*

Assume **constant ROE and plowback ratio:**

$$g = ROE \times \textbf{Plowback ratio}$$

4.6 Present Value of Growth Opportunities (PVGO)

If the firm didn't reinvest profits back into the company, would it be worth more? The PVGO helps answer that.

Let

$P_{\rho=\rho}$ be the price of the firm normally, when earnings are plowed back.

$P_{\rho=0}$ be the price of the firm with no plowback, hence no growth in earnings

Hence:

$$P_{\rho=\rho} = \frac{D}{r_s - g} \text{ and } P_{\rho=0} = \frac{EPS}{r_s}$$

We define:

$$PVGO = P_{\rho=\rho} - P_{\rho=0}$$

Substituting in $P_{\rho=\rho}$ and $P_{\rho=0}$ we get:

$$PVGO = \frac{D}{r_S - g} - \frac{EPS}{r_S}$$

5 The Capital Asset Pricing Model (CAPM)

5.1 Introduction and Motivation

When selecting a portfolio of assets, we have two objectives:

1. maximise **RETURNS = geometric mean historical returns**
2. minimise **RISK = variance/st.dev.**

5.2 Diversification & Beta

The crucial point that the derivation in the appendix will use is the fact that diversification raises the overall return/variance ratio.

R_i = return on asset i
R_P = return on portfolio
R_M = market return

Mathematically:

$$Var(R_P) = \sum_{j=1}^{N} \sum_{i=1}^{N} w_i w_j Cov(R_i, R_j) = \sum_{i=1}^{N} w_i^2 Var(R_i) + 2 \sum_{i \neq j}^{N} w_i w_j Cov(R_i, R_j)$$

Beta
Beta is the contribution of asset i to the overall portfolio variance, which we can write as:

$$\beta_i = \frac{Cov(R_i, R_P)}{Var(R_P)}$$

It can also be viewed as the slope coefficient in a linear regression of R_i on R_P.

5.3 The CAPM Equation*

$$\mathbb{E}[R_i] = R_f + \beta_i \left(\mathbb{E}[R_M] - R_f \right) \qquad ; \beta_i = \frac{Cov(R_i, R_M)}{Var(R_M)}$$

The CAPM gives us an expected return to discount cashflows based on their beta risk.

5.4 Equilibrium

Assumptions
All investors agree on expected returns of stocks, covariances and variances. The market is frictionless.

Demand
All investors hold risky assets in proportions as dictated by the tangency portfolio

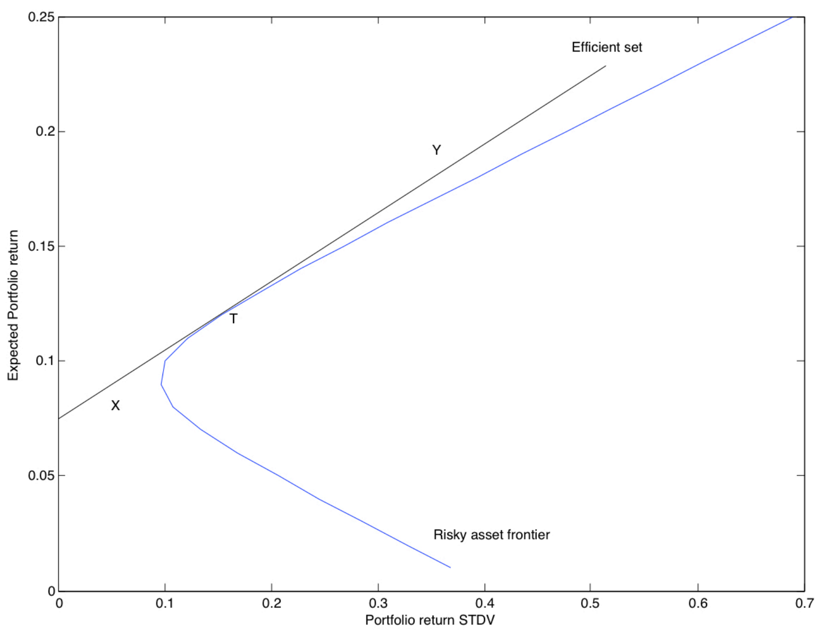
Supply
The value of stocks issued is the supply of risky assets, the market portfolio represents the market capitalisation weighted portfolio.

In equilibrium, the market portfolio = the tangency portfolio

5.5 The Capital Markets Line (CML)

The CML plots expected return against standard deviation.

The Capital Markets Line shows the set of portfolios consisting the risk-free rate and the market portfolio, which are the "efficient set", i.e. the best risk-reward ratio portfolios.

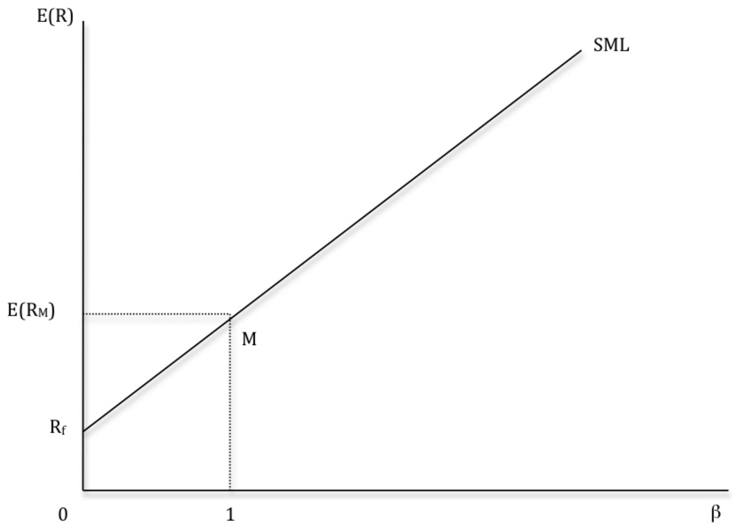


(figures from FM213 slides, Peng, 2022)

5.6 The Security Markets Line (SML)

The Security Markets Line plots expected return against beta.

The CAPM says that the only reason why stock returns are different is due to differing betas. Hence if one knows the beta of an asset, one can calculate its expected return under the CAPM using the SML (or the CAPM formula).



(figures from FM213 slides, Peng, 2022)

5.7 Using CAPM for Valuation

Valuation of stocks:
Think of a stock as just a claim to a stream of future dividends. How should we value the stock? Compute the NPV of the future estimated dividend stream. What discount rate should we use? Use the CAPM expected return for a company. Why? Because the CAPM tells you what return you should expect from a company with that level of risk – it gives a risk-adjusted discount rate.

Valuation of projects:
Similar intuition to the above. For an equity financed firm, the CAPM expected return is the return that one should require any project that the firm takes on to earn. Thus use it as the discount rate in a NPV calculation.

5.8 Empirical Failures

Factors other beta seem to be able to explain expected returns better than the CAPM. These are:

1. **Size**
   Smaller companies tend to have higher expected return than bigger firms, given the same beta.

2. **Value**
   Mean returns on stocks with high book-to-market tend to be larger than those on low book-to-market stocks (holding β equal)

5.9 Alternative Models

Arbitrage Pricing Theory (APT)/Fama-French 3 Factor Model/Multi-Factor Models

A common model used is the Fama-French 3 factor model that adds another two sources of risk (size and value) to beta. It is an example of APT and a multi-factor pricing model.

$$E(R_j) = R_f + \beta_{M,j}(\mathbb{E}[R_M] - R_f) + \beta_{S,j}\mathbb{E}[R_{SML}] + \beta_{V,j}\mathbb{E}[R_{HML}]$$

1. **Size**

   $\beta_{S,j}$ is the stock's beta to a size portfolio.

   The size portfolio (called SML) is long small stocks and short large stocks and has expected return equal to $\mathbb{E}[R_{SML}]$

2. **Value**

   $\beta_{V,j}$ is the stock's beta to a value portfolio.

   The value portfolio (called HML) is long stocks with low market value relative to book value and short stocks with high market value relative to book value and has expected return equal to $\mathbb{E}[R_{HML}]$

6 Efficient Markets Hypothesis (EMH)

6.1 Motivation

Is it possible to make "genuine" returns in financial markets?
Note the use of the word "genuine".

What can we class as genuine?
- From CAPM we know that if one takes leverage, or more risk, then their expected return is greater.
- We also know that stock performance can be modelled as random, well then we also need to ascertain whether superior returns are just a random occurrence.

Hence, in addition to the original question, we also need to answer how we can measure "superior" returns. Are returns due to more risks taken or

Another point to start the discussion on efficient markets is that if everyone could make "superior" returns, then it doesn't seem that "superior" after all - in fact isn't it just free money? Surely it must conflict with our assumptions of no-arbitrage* in asset pricing? (*definition in chapter 7)

6.2 Definitions and Assumptions

The information set, Ω :

The available information to investors to try and forecast returns on securities

Definitions of Efficiency

Jensen's definition of Efficiency (1978):

> "A market is said to be informationally efficient with respect to an information set Ω , if an investor cannot make economic profits by trading on the information contained in Ω."

Malkiel's definition of Efficiency (1992):

> "A capital market is said to be efficient if it fully and correctly reflects all relevant information in determining security prices, i.e. with respect to Ω."
>
> This implies that it is impossible to make economic profits by trading on the basis of Ω.

What conditions allow efficiency to hold?

3. Rationality of market participants
4. or Balanced deviations from rationality
5. or Dominance of rational professional investors

Any one of these is sufficient for efficiency to hold.

6.3 Types of Efficiency

Roberts (1967) came up with the following classification for the strength of efficiency in a market.

Strong Form
All **public and private** information is incorporated into the price

Semi-strong Form
All **public** information is incorporated into the price

Weak Form
All **past prices** incorporated into the price

Another notion of efficiency to consider is:

Operational Efficiency
How cheap, easy and quick it is to trade on the market.

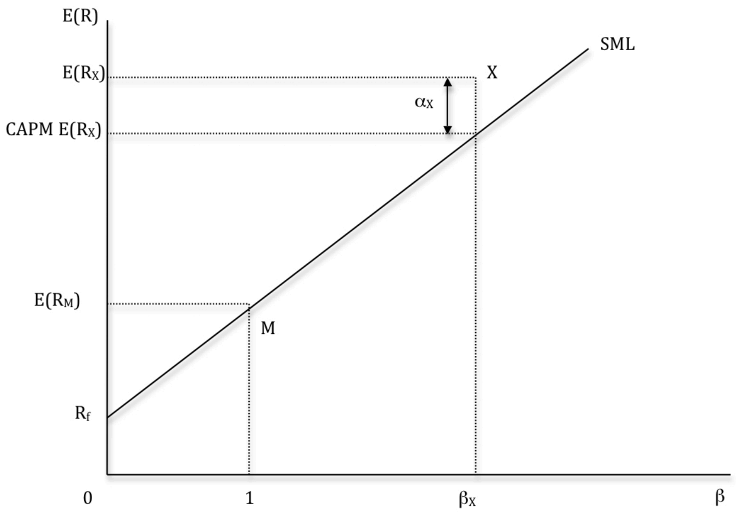
Informational Efficiency
Whether securities are correctly priced given the available information.

6.4 Alpha/Abnormal Returns & the CAPM

We define alpha as excess returns over what the CAPM predicts:

$$\alpha = [R_P] - [R_f + \beta_P(R_M - R_f)]$$

Graphical View:



(from FM213 slides, Peng, 2022)

Trading Strategy
If a stock has above expected returns, long the stock as the price should rise until returns are lower.

And vice versa, if a stock produces below expected returns, the price should fall until returns are higher, hence short the stock.

6.5 Joint Hypothesis problem

All tests of efficiency are a test of a joint hypothesis containing 2 parts;

1. The market under analysis is informationally efficient.

2. The researcher knows and uses the correct asset pricing model to generate abnormal returns.

If we reject the null hypothesis it can be because either (or both) of the parts of the hypothesis are invalid. This creates uncertainty when interpreting any test results. For example:

1. We find empirically that the strategy of:
   - buying stocks on the announcement of new dividends
   - selling stocks on the announcement of a dividend cut

   returns alpha.

2. One way to interpret this is that it leads us to reject the notion of efficiency in this market.

3. A second interpretation is that the expected returns we computed using the CAPM are not accurate. They are based on the CAPM and the CAPM is not the right model for expected returns.

Thus, the joint-hypothesis problem clouds our ability to interpret results from efficiency tests.

6.6 A Fair Game

An implication of the efficiency is that if a market is efficient with respect to Ω, then **future abnormal returns appear to be entirely random from the perspective of an individual who has the information set given by Ω.**

The EMH implies that excess returns are a fair game, that is:

$$\mathbb{E}[\alpha_{t+1} \quad \Omega_t] = 0$$

This equation says the expected alphas are 0 in the next period, given the information set today.

From an econometric perspective, if the slope of:

$$\alpha_t = c_0 + c_1 X_{t-1} + \epsilon_t$$

is statistically zero, then this suggests efficiency, else it implies X_{t-1} predicts α_t

NB. X_t is a prediction variable in Ω_t

6.7 Summary, EMH

⇒ efficiency requires all information to be quickly and accurately reflected in prices.

⇒ active trading is a fruitless pastime as no-one can beat the market.

⇒ under the CAPM, α's are on average zero and are not forecastable.

7 Futures, Forward Valuation

7.1 Derivatives Introduction & Examples

A derivative security is a security for which the payoff is governed entirely by the value of one or more underlying assets.

Standard (Vanilla) Examples

- Futures
- Forwards
- Call and Put Options
- Swaps

These can be based on commodities, stocks, bonds, currencies, anything really…

Uses

- Hedging risk
- Making specific bets
- Create leverage

7.2 Valuation Principles

Arbitrage
An arbitrage opportunity is a strategy with no cost greater than 0, but with potential payoff greater than 0. In other words, a free lunch.

Law of One Price (LOOP)
From the absence of arbitrage assumption, it implies the law of one price. **It says that two portfolios with identical payoffs, have the same price.**

Law of Payoff Dominance
Also from the absence of arbitrage assumption, **if one portfolio guarantees a better payoff than another, it must have a higher price.**

7.3 Forward Definition

A forward contract is an agreement to buy/sell a certain quantity of an asset for a pre-specified price at a particular future date. Forwards are traded OTC (over the counter).

The cost to enter a forward option today is defined to be 0, hence the value of the forward contract at inception is 0.

7.4 Forward Price (no yield)*

Let

$K =$ forward price/delivery price at maturity
$S_t =$ stock price at time t, $S_T =$ stock price at maturity
$r_T =$ T period spot rate

If we can replicate the payoff at maturity with another portfolio, then they must have the same cost at all times. This is the idea behind the derivation in the appendix, where the forward is replicated with a stock and a bond.

The forward price is given by:

$$K = S_0(1 + r_T)^T$$

OR

With continuous compounding at rate r,

$$K = S_0 e^{rT}$$

7.5 Forward Price (known income)*

Let:

$I =$ sum of PV of all income/dividends accrued to the asset holder during the forwards life, hence $I = \sum_{t=0}^{T} PV(C_t)$

The forward price is just:

$$K = (S_0 - I)(1 + r_T)^T$$

OR

$$K = (S_0 - I)e^{rT}$$

7.6 Forward Price (known yield)*

In addition to above, let

$q =$ known yield on asset, can be currency (interest rate) or stock (dividend yield) or negative if it is a convenience yield for commodities

then

$$K = S_0 \left(\frac{1 + r_T}{1 + q_T} \right)^T$$

OR

$$K = S_0 e^{(r-q)T}$$

8 Option Strategies

8.1 Option Contract Definition & Terminology

The right to buy an asset, at a predetermined price (not obligation).

European options can only be exercised at expiration/maturity, whereas **American options** can be exercised at any time.

We summarise the various terminology used briefly below:

Term	Symbol	Definition
Underlying asset price	S_t	The price of the asset that the option is based on, at time t
Maturity/Expiration date	$t = T$	How long the option lasts for
Strike Price	K	The price agreed that can option can be exercised at
Call option	Value at time t $= c_t$	Option to buy a stock/asset
Put option	Value at time t $= p_t$	Option to sell a stock/asset
American option		Option can be exercised at any time
European option		Option can only be exercised at maturity

8.2 Option Positions

First of all we can either choose a call or put option. There is also another choice of whether to be long or short a call/put option. This means **we are selling someone the option to buy a stock (short call), in return we get a guaranteed payment, the price of the option.**

LONG VS SHORT = BUY VS SELL THE OPTION

PUT VS CALL = BUY VS SELL THE STOCK

NB. without explicit mention of short, assume long position.

8.3 Option Payoff Formulas

The payoff from holding a call option is as followed:

CASE 1: **IN THE MONEY** where $S_T > K$
$\Rightarrow$ we exercise the option making return of $S_T - K$

CASE 2: **AT THE MONEY** where $S_T = K$
$\Rightarrow$ the option makes return of 0, as $S_T - K = 0$

CASE 3: **OUT OF THE MONEY** where $S_T < K$
$\Rightarrow$ we don't exercise the option, as $S_T - K < 0$

Hence for a call option we can summarise the payoff mathematically as

call option payoff at expiration $= \max[S_T - K, 0]$ **or** $(S_T - K)^+$

And for a put option it is:

put option payoff at expiration $= \max[K - S_T, 0]$ **or** $(K - S_T)^+$

The payoff for the short version of each is **just the negative**:

short call option payoff at expiration $= -\max[S_T - K, 0]$ **or** $-(S_T - K)^+$

short put option payoff at expiration $= -\max[K - S_T, 0]$ **or** $-(K - S_T)^+$

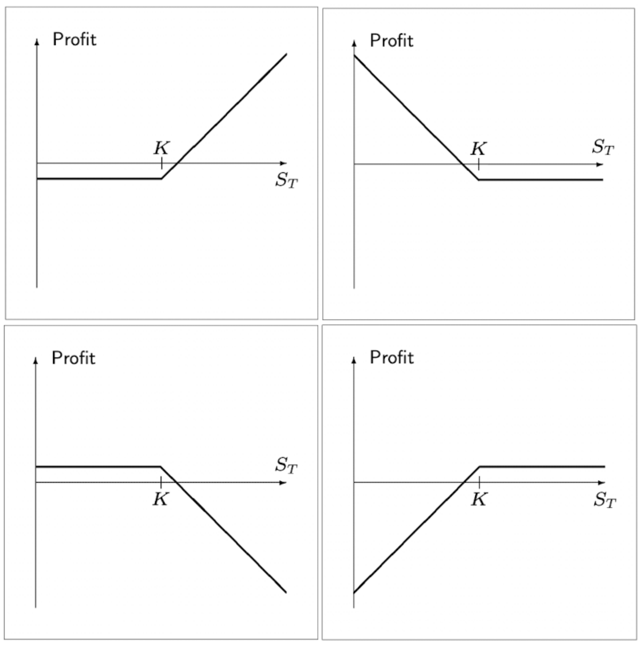
8.4 Option Payoff Graphs

(The intercept is just the FV of the option price, i.e. the insurance cost)

CALL **PUT**

LONG

SHORT



(figures from FM213 slides, Peng, 2022)

8.5 Options as Leverage

It may seem that options are safer than stocks because there is a lower bound.

However this is wrong because options contain implicit leverage. You only need to pay for one option to control typically 100 units of stock.

Also, there is a cost to options, which is the premium. If the stock price falls below a certain value, then you lose all your money - the premiums/option price paid.

Imagine the following scenario:

$$S_0 = 35 \text{ and there exists } c_0 = 5, K = 30$$

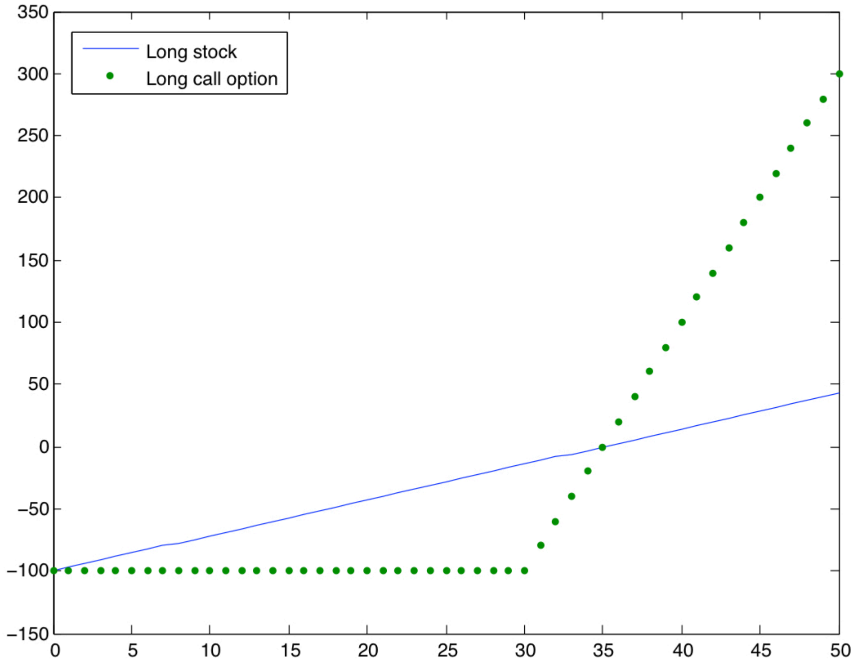
We construct two portfolios, one of **options**, one of **stocks shown below:**

$$\Pi_1 : S_0$$
$$\Pi_2 : 7c_0$$

They have the same cost, =35. Now plot the return of the portfolio varying S_t from 0 to 50

$$1 + R_{\Pi_1} = \frac{S_t}{S_0}$$

$$1 + R_{\Pi_2} = \frac{7\max[S_t - K, 0]}{7c_0} = \frac{\max[S_t - K, 0]}{c_0}$$



Portfolio value against stock price
(from FM213 slides, Peng, 2022)

8.6 Bull/Bear spreads

Bull/Bear spreads are used for a **mildly bullish/bearish view**, i.e. doesn't believe stock will move an extreme amount.

In return for **sacrificing extra returns in the extreme case**, the investor gets to **pay a lower premium**, hence breaks even at expiration with a lower stock movement.

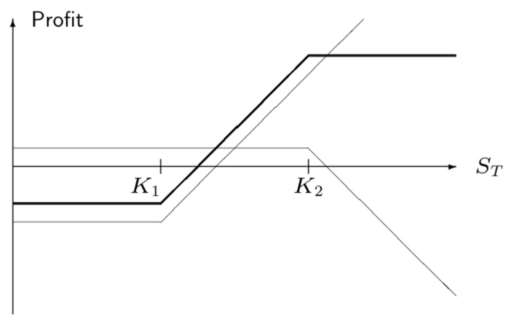
There are two types:
Bull Spread: mildly bullish, i.e. bet stock moves up slightly

Bear Spread: mildly bearish, i.e. bet stock moves down slightly

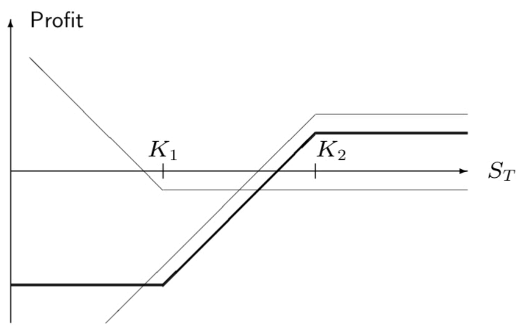
Both of these positions can constructed using either a combination of calls or a combination of puts, which is demonstrated below

BULL SPREAD

Using Calls

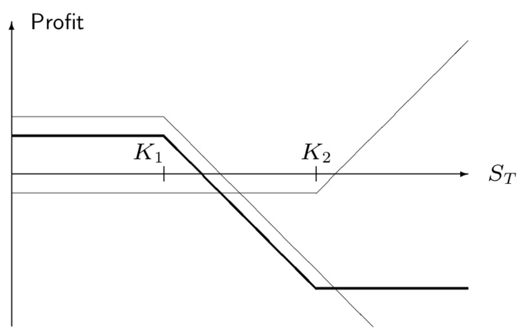


Using Puts

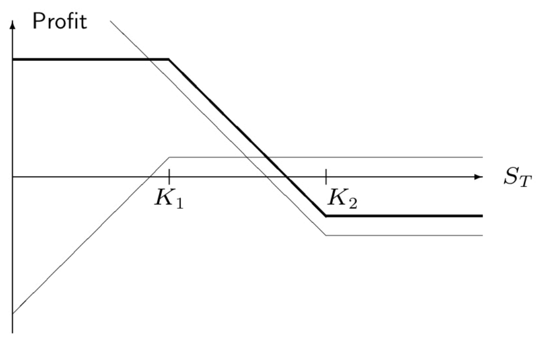


BEAR SPREAD

Using Calls



Using Puts



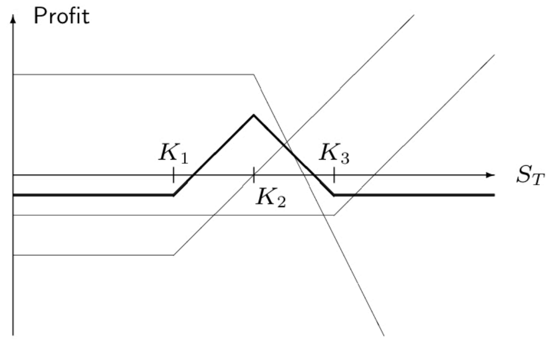
(figures from FM213 slides, Peng, 2022)

8.7 Butterfly spread

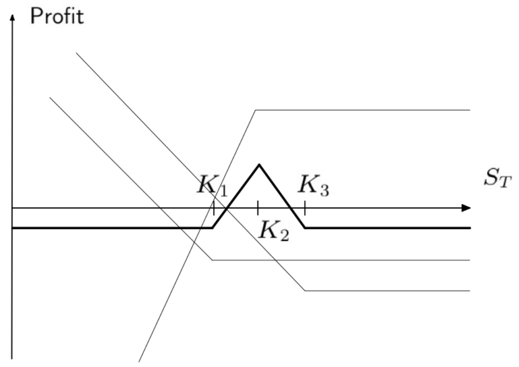
Butterfly spreads are used when an **investor doesn't believe the stock will move a lot in either direction, i.e. not very volatile.**

The investor takes profit when the stock does not move much, but incurs a loss in the case of extreme movement, shown below:

Using Calls
Long 1 Call @K_1
Short 2 Calls @K_2
Long 1 Call @K_3



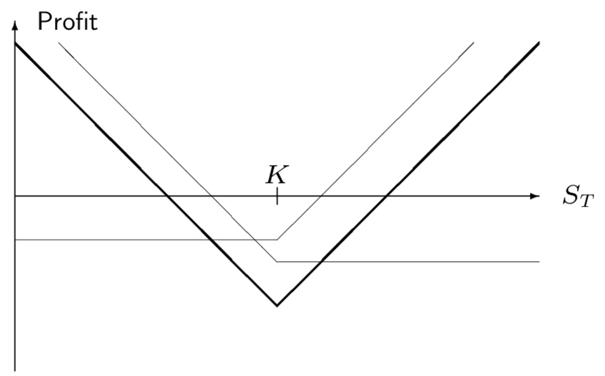
Using Puts
Long 1 Put @K_1
Short 2 Puts @K_2
Long 1 Put @K_3



8.8 Straddle

In contrast to the butterfly spread above, **if an investor believes there will be high volatility ahead, they can enter into a straddle position**, which takes profit if the stock price moves a lot in either direction, but incurs a loss if the stock price moves little.

Long Straddle



(figures from FM213 slides, Peng, 2022)

9 Option Pricing

9.1 Binomial Option Model - Setup

Stock Properties

Let

S_0 = current stock price
u = factor up
d = factor down

and $u > d \geq 0$

The stock price at maturity, T can either be "up" or "down":

$$S_T = \begin{cases} S_T^u = uS_0 & \textbf{if "up"} \\ S_T^d = dS_0 & \textbf{if "down"} \end{cases}$$

Option Properties

Let

c_0 = value of call option today
c_T^u = "up" payoff at maturity, T
c_T^d = "down" payoff at maturity, T

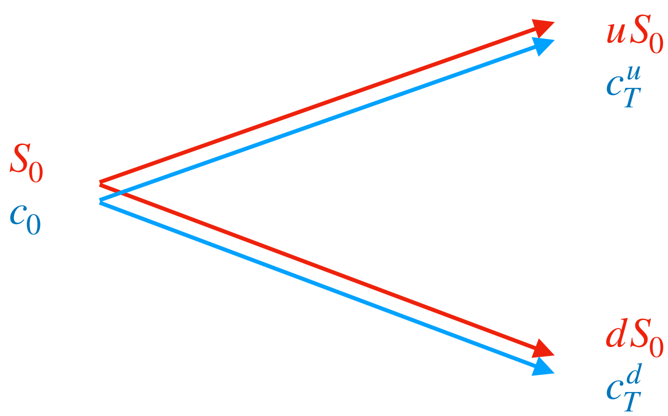
and K = strike price/exercise price
r = risk free rate adjusted for one period of option

Hence:

$$c_T = \begin{cases} c_T^u & \textbf{if "up"} \\ c_T^d & \textbf{if "down"} \end{cases}$$

(NB. From previous chapter we actually know that $c_T = \begin{cases} c_T^u = \max[uS_0 - K, 0] \\ c_T^d = \max[dS_0 - K, 0] \end{cases}$, but we can ignore this right now.)

Binomial tree diagram:



Our objective: Find the price of option, c_0

9.2 Binomial Option Model, Method 1 - Pricing by Replication

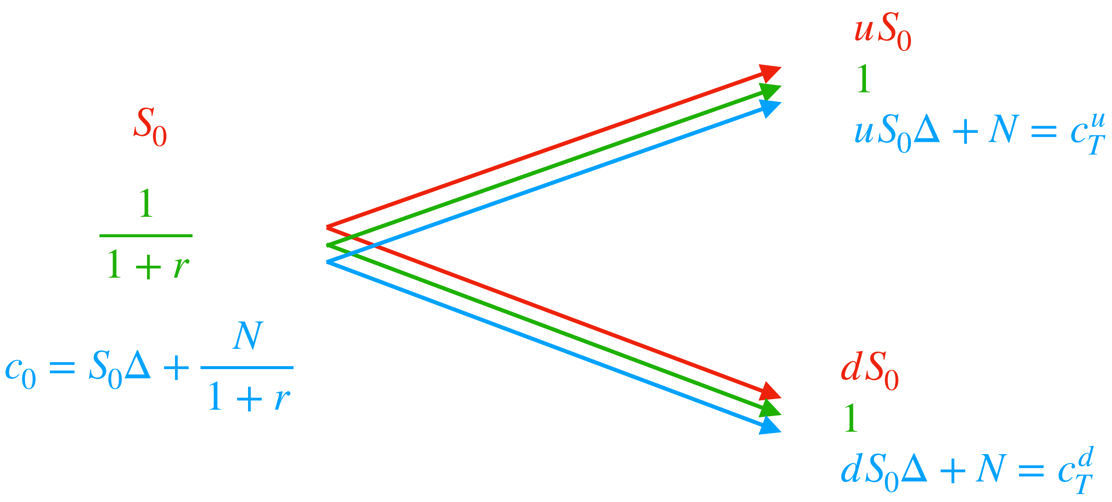
Imagine a portfolio of:

Δ number of stocks S_0 , and

N number of zero coupon bonds with maturity T and $FV = 1$

The payoffs of the portfolio expiration are given below on the tree:

Key
red : **stock** payoff
green : **bond** payoff
blue : **portfolio / option** payoff



We equate in blue the
1) **payoffs of the option and the replicating portfolio**
2) and the **cost of the option and the replicating portfolio**

The solution to the two blue equations on RHS are:

$$\Delta = \frac{c_u - c_d}{S_0(u-d)}$$

$$N = \frac{uc_d - dc_u}{u-d}$$

Which we plug into the option price:

$$c_0 = S_0\Delta + \frac{N}{1+r}$$

9.3 Binomial Option Model, Method 2 - Delta Hedging

Similar to above, we can construct a portfolio, consisting of stocks and options **to create a risk-less payoff**, like a bond. Since the payoff is risk-less, we can **discount it at the risk free rate** to help find the option price.

Imagine a portfolio of:

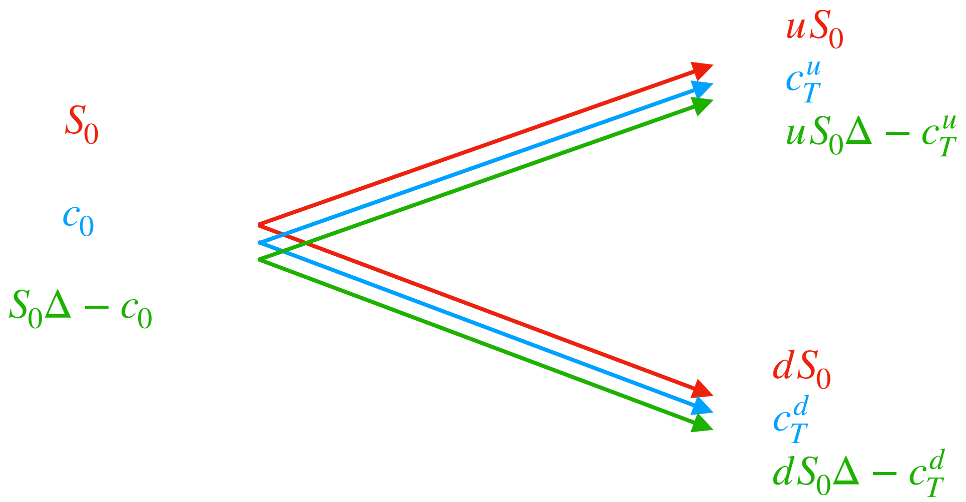
- **Δ number of stocks** S_0
- **-1 call option,** c_0

Key

red : **stock** payoff

blue : **option** payoff

green : **portfolio** payoff



In order for risk-less payoff, the two branches must be the same value, $uS_0\Delta - c_T^u = dS_0\Delta - c_T^d$ which gives again:

$$\Delta = \frac{c_u - c_d}{S_0(u-d)}$$

We also know the return is risk-free, hence when discounted, it should equal its price.

$$S_0\Delta - c_0 = \frac{1}{1+r}[uS_0\Delta - c_T^u]$$

Substituting Δ gives us

$$c_0 = \frac{1}{1+r}[qc_T^u + (1-q)c_T^d]$$

where

$$q = \frac{(1+r) - d}{u-d}$$

9.4 Binomial Option Model, Method 3 - Risk Neutral Measure

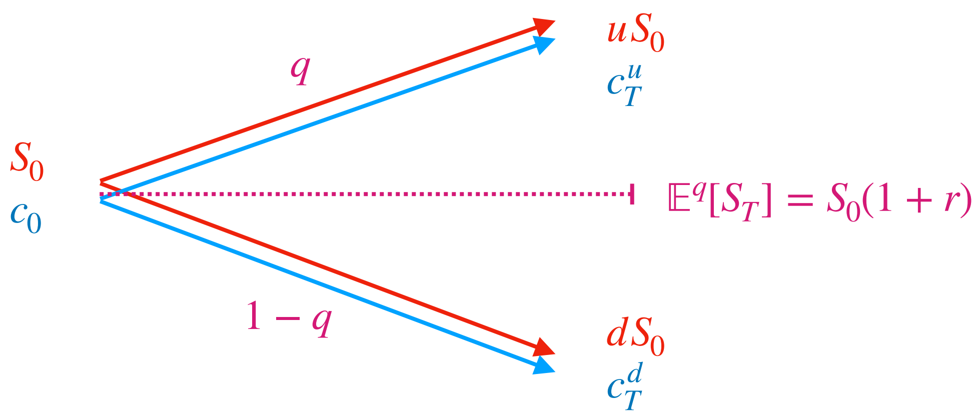
If all investors were **risk-neutral, i.e indifferent to risk,** then **any asset would be discounted at the risk free rate,** since we don't care about risk.

Hence the expected return on any asset at expiration is $(1 + r)$ times its original value.

$$\mathbb{E}^q[S_T] = S_0(1 + r)$$

We can then adjust real world probabilities of events to incorporate riskiness shown below

NB. It may sound complex but is the easiest method of the three to remember. Just follow the two steps below.



Step 1 - Find q under risk neutral measure

$$q \cdot uS_0 + (1 - q) \cdot dS_0 = S_0(1 + r)$$

Step 2 - Find expected value of option using q above, then discount it

$$c_0 = \frac{1}{1 + r}\left[q \cdot c_T^u + (1 - q) \cdot c_T^d\right]$$

9.5 Option Price Bounds*

Lower Bounds

1) $c_0 \geq 0$ because the payoff at expiration is always bigger than 0,
$c_T = \max[0, S_T - K]$

2) $c_0 \geq S_0 - Ke^{-rT}$ *(derivation in appendix)

Upper Bound

3) $c_0 \leq S_0$ because of no arbitrage

9.6 Put-Call Parity (no dividends)*

Like before, let

$c_t =$ value of call option at time t
$p_t =$ value of put option at time t
$S_t =$ stock price at time t
$K =$ strike price

The put-call parity formula is:

$$S_t + p_t = c_t + Ke^{-r(T-t)}$$

9.7 The Black-Scholes-Merton Model*

In the BS model,

1) **Time passes continuously** instead of in discrete intervals as in the binomial model.
2) **Returns are distributed normally** instead of only two outcomes.

A model for stock returns
The second assumption is very important to the BS model below. The percentage change in stock price, aka return is assumed to be distributed normally.

$$\frac{\delta S}{S} \sim N(\mu \delta t, \sigma^2 \delta t)$$

where

$\mu =$ annualised mean returns of stock
$\sigma =$ annualised volatility of stock returns

Call Option
The price of a call is given by:

$$c_0 = S_0 \cdot \Phi(d_1) - Ke^{-rT} \cdot \Phi(d_2)$$

where

$$d_1 = \frac{\ln(\frac{S_0}{K}) + T(r + \frac{\sigma^2}{2})}{\sigma\sqrt{T}} \quad \textbf{AND} \quad d_2 = d_1 - \sigma\sqrt{T}$$

NB. $\Phi(x)$ is cumulative normal distribution

Put Option*
Using the put-call parity relationship in 9.6, we can find the price of a put (the derivation for this is in appendix).

$$p_0 = Ke^{-rT} \cdot \Phi(-d_2) - S_0 \cdot \Phi(-d_1)$$

again where

$$d_1 = \frac{\ln(\frac{S_0}{K}) + T(r + \frac{\sigma^2}{2})}{\sigma\sqrt{T}} \quad \textbf{AND} \quad d_2 = d_1 - \sigma\sqrt{T}$$

9.8 Option Price Factors

How does the price of an option change with respect to the inputs?

increase in	**call price, c_0**	**put price, p_0**
S_0, stock price	↑, **increase**	↓, **decrease**
K, strike price	↓, **decrease**	↑, **increase**
σ, return volatility	↑, **increase**	↓, **decrease**
T, expiration date	↑, **increase**	↓, **decrease**
r, risk-free rate	↑, **increase**	↓, **decrease**

9.9 Historical vs Implied Volatility (IV)

How can we get the value for σ, volatility? We can:

1) **Estimate from historical prices** by calculating standard deviation of daily returns then annualising that number by multiplying by $\sqrt{252}$, which is the number of trading days

2) **Estimate by backing out the volatility parameter from an option price already traded** in the market, on that same asset. This is called Implied Volatility (IV)

Appendix

The relevant chapter is the number after the A in the sub-headers

A.2.1 Perpetuity (w/o growth)

By definition, the present value of a perpetuity without growth is

$$PV(P) = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t}$$

Note $PV(P)$ above is a geometric sum to infinity with first term $\frac{C}{1+r}$ and common ratio $\frac{1}{1+r}$.

Hence using the standard sum to infinity formula, $\frac{\text{first term}}{1 - \text{common ratio}}$, we get

$$PV(P) = \frac{\frac{C}{1+r}}{1 - \frac{1}{1+r}}$$

Now multiply top and bottom by $1 + k$,

$$\Rightarrow PV_0(P) = \frac{C}{(1+r) - 1}$$

$$\Rightarrow PV_0(P) = \frac{C}{r}$$

A.2.2 Perpetuity (w growth)

By definition, the present value of a perpetuity with growth is

$$PV(P_g) = \sum_{t=1}^{\infty} \frac{C(1+g)^{t-1}}{(1+r)^t}$$

Note $PV(P_g)$ is another infinite geometric sum with first term $\frac{C}{1+r}$ and common ratio $\frac{1+g}{1+r}$.
Hence using the sum to infinity formula $\frac{\text{first term}}{1 - \text{common ratio}}$, we get

$$PV_0(P) = \frac{\frac{C}{1+r}}{1 - \frac{1+g}{1+r}}$$

Now multiply top and bottom by $1 + r$,

$$\Rightarrow PV(P) = \frac{C}{(1+r) - (1+g)}$$

$$\Rightarrow PV(P) = \frac{C}{r-g}$$

A.2.3 Annuity (w/o growth)

By definition, the present value of an annuity without growth is

$$PV(A) = \sum_{t=1}^{T} \frac{C}{(1+r)^t}$$

Method 1:

We can view the annuity as the difference of two perpetuities (w/o growth), one starting with payments at year 1 to infinity denoted P_0, subtract another with payments starting at T+1 til infinity denoted P_T, both discounted to present value.

Hence by definition above,

$$PV(P_0) = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

$$PV(P_T) = PV_0\left(\sum_{t=T+1}^{\infty} \frac{C}{(1+r)^t}\right) = PV\left(\frac{C}{r}\right) = \frac{1}{(1+r)^T} \cdot \left(\frac{C}{r}\right)$$

And so,

$$PV(A) = PV(P_0) - PV(P_T)$$

$$\Rightarrow PV(A) = \left(\frac{C}{r}\right) - \frac{1}{(1+r)^T} \cdot \left(\frac{C}{r}\right)$$

$$\Rightarrow PV(A) = \frac{C}{r}\left[1 - \frac{1}{(1+r)^T}\right]$$

Method 2:

We can also use the finite sum formula.

Given a geometric series: $\sum_{n=1}^{N} ar^{n-1}$, which has first term a, common ratio r, the sum to N is given by: $a\left(\frac{1-r^N}{1-r}\right)$

The annuity has first term, $\frac{C}{1+r}$ and common ratio, $\frac{1}{1+r}$:

Hence the sum is:

$$\frac{C}{1+r}\left(\frac{1-\frac{1}{1+r}^{T}}{1-\frac{1}{1+r}}\right)$$

Multiply top and bottom of fraction by $1+r$

$$\equiv C\left(\frac{1-\frac{1}{1+r}^{T}}{(1+r)-1}\right)$$

$$\equiv \frac{C}{r}\left[1-\left(\frac{1}{1+r}\right)^{T}\right]$$

A.2.4 Annuity (w growth)

By definition, the present value of an annuity with growth is

$$PV(A_g) = \sum_{t=1}^{T} \frac{C(1+g)^{t-1}}{(1+r)^t} = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right]$$

Again we can view the annuity as the difference of two perpetuities (w growth), one starting with payments at year 1 to infinity denoted P_0, subtract another with payments starting at T+1 til infinity denoted P_T, both discounted to present value.

Hence by definition above,

$$PV(P_0) = \sum_{t=1}^{\infty} \frac{C(1+g)^{t-1}}{(1+r)^t} = \frac{C}{r-g}$$

However $PV_0(P_T)$ is slightly more complicated due to growth. The first cash payment becomes $C(1+g)^T$. Hence,

$$PV(P_T) = PV\left(\sum_{t=T+1}^{\infty} \frac{C(1+g)^T(1+g)^{t-1}}{(1+r)^t}\right)$$

$$\Rightarrow PV(P_T) = PV\left(\frac{C(1+g)^T}{r-g}\right)$$

$$\Rightarrow PV(P_T) = \frac{1}{(1+r)^T} \cdot \left(\frac{C(1+g)^T}{r-g}\right) = \frac{(1+g)^T}{(1+r)^T} \cdot \left(\frac{C}{r-g}\right)$$

And so,

$$PV(A) = PV(P_0) - PV(P_T)$$

$$\Rightarrow PV(A) = \left(\frac{C}{r-g}\right) - \frac{(1+g)^T}{(1+r)^T} \cdot \left(\frac{C}{r-g}\right)$$

$$\Rightarrow PV(A) = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right]$$

Method 2:

We can also use the finite sum formula.

Given a geometric series: $\sum_{n=1}^{N} ar^{n-1}$, which has first term a, common ratio r, the sum to N is given by: $a\left(\frac{1-r^N}{1-r}\right)$

The annuity has first term, $\frac{C}{1+r}$ and common ratio, $\frac{1+g}{1+r}$:

Hence the sum is:

$$\frac{C}{1+r}\left(\frac{1-\frac{1+g}{1+r}^{T}}{1-\frac{1+g}{1+r}}\right)$$

Multiply top and bottom of fraction by $1+r$

$$\equiv C\left(\frac{1-\frac{1+g}{1+r}^{T}}{(1+r)-(1+g)}\right)$$

$$\equiv \frac{C}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^{T}\right]$$

A.2.6 Continuous Compounding

We start with definition of effective annual rate for 1 period/year

$$1 + EAR = \left[1 + \frac{r}{N}\right]^N$$

Take the limit of both sides

$$\Rightarrow \lim_{N\to\infty} [1 + EAR] = \lim_{N\to\infty} \left[1 + \frac{r}{N}\right]^N$$

substitute $N = mr$

$$\Rightarrow \lim_{N\to\infty} [1 + EAR] = \lim_{m\to\infty} \left[1 + \frac{1}{m}\right]^{mr}$$

substitute definition of $e = \lim_{x\to\infty} \left[1 + \frac{1}{x}\right]^x$ and hence we get

$$\Rightarrow \lim_{N\to\infty} [1 + EAR] = e^r$$

A.2.7 Real and Nominal Rates (Fisher Relationship)

Let

π = inflation rate
i = nominal interest rate
r = real interest rate

The real interest rate is given by:

$$[1 + r] = \frac{[1 + i]}{[1 + \pi]}$$

$$\Rightarrow \ln[1 + r] = \ln[1 + i] - \ln[1 + \pi]$$

When x is small, $\ln(1 + x) = x$, hence

$$\Rightarrow r = i - \pi$$

A.3.3 Macaulay Duration

Let $D =$ Macaulay Duration, the negative of the elasticity of bond price with respect to yield.

Hence by definition, the elasticity is:

$$D = -\frac{\%\,\Delta P}{\%\,\Delta(1+y)}$$

$$\Rightarrow D = -\frac{\frac{\Delta P}{P}}{\frac{\Delta(1+y)}{(1+y)}}$$

$$\Rightarrow D = -\frac{(1+y)}{P}\frac{\Delta P}{\Delta(1+y)}$$

$$\Rightarrow D = -\frac{1+y}{P}\frac{dP}{d(1+y)}$$

Now let's work out what $\frac{dP}{d(1+y)}$ is. Start with relationship of bond price and yield:

$$P = \sum_{t=1}^{T}\frac{C_t}{(1+y)^t} = \sum_{t=1}^{T} C_t(1+y)^{-t}$$

$$\Rightarrow \frac{dP}{d(1+y)} = \sum_{t=1}^{T} -t\frac{C_t}{(1+y)^{t+1}}$$

Hence we get Macaulay duration is

$$\Rightarrow D = -\frac{1+y}{P}\frac{dP}{d(1+y)} = \frac{(1+y)}{P}\cdot\sum_{t=1}^{T} t\frac{C_t}{(1+y)^{t+1}}$$

$$\Rightarrow D = \frac{1}{P}\sum_{t=1}^{T} t\cdot\frac{C_t}{(1+y)^t}$$

A.3.4 Modified Duration

Let D^M **denote Modified Duration**, representing the **percentage change in price** for a given change in yield.

By definition, (using a first order Taylor series):

$$\% \Delta P = \frac{\Delta P}{P} \approx - D^M \cdot \Delta y$$

Hence we can express modified duration as

$$\Rightarrow D^M = - \frac{\frac{dP}{dy}}{P}$$

Using the derivation in the previous section for $\frac{dP}{d(1+y)}$ which $= \frac{dP}{dy}$,

$$\Rightarrow D^M = - \frac{1}{P} \sum_{t=1}^{T} - t \frac{C_t}{(1+y)^{t+1}}$$

which we can see means Modified Duration and Macaulay Duration are linked by

$$D^M = \frac{D}{1+y}$$

A.4.2 Stocks, Basic Valuation Theory

Let:

S_t = stock price at time t
D_t = dividend/cash flow at time t
$\bar{r}_S$ = average risk-adjusted discount rate/return for the specific stock

Also **assume stock returns are equal every year.**

Start with year 1: The return in year 1 is the sum of the

abs. return = (dividends in year one) + (capital gains - i.e. change in stock price)

we can write this as:

$$\bar{r}_S P_0 = \mathbb{E}[D_1] + (\mathbb{E}[P_1] - P_0)$$

then continue to year 2 and beyond:

$$\bar{r}_S P_1 = \mathbb{E}[D_2] + (\mathbb{E}[P_2] - P_1)$$
$$\cdots$$
$$\bar{r}_S P_t = \mathbb{E}[D_t] + (\mathbb{E}[P_t] - P_{t-1})$$

We start the derivation now for P_0 :

Rearrange for P_0 from the first equation

$$P_0 = \frac{\mathbb{E}[D_1] + \mathbb{E}[P_1]}{(1 + \bar{r}_S)}$$

But we know from second equation that P_1 is just

$$\mathbb{E}[P_1] = \frac{\mathbb{E}[D_2] + \mathbb{E}[P_2]}{(1 + \bar{r}_S)}$$

Hence combining them we get:

$$P_0 = \frac{\mathbb{E}[D_1]}{(1 + \bar{r}_S)} + \frac{\mathbb{E}[D_2] + \mathbb{E}[P_2]}{(1 + \bar{r}_S)^2}$$

if we continue the iteration until $t = T$, then we get:

$$P_0 = \frac{\mathbb{E}[D_1]}{(1 + \bar{r}_S)} + \frac{\mathbb{E}[D_2]}{(1 + \bar{r}_S)^2} + \ldots \frac{\mathbb{E}[D_T] + \mathbb{E}[P_T]}{(1 + \bar{r}_S)^T}$$

Since we can assume stock lives forever,
so take limit as $T \to \infty$, also we can assume that $\lim_{T\to\infty} \left[\frac{\mathbb{E}[P_T]}{(1 + \bar{r}_S)^T} \right] = 0$

Hence we get:

$$P_0 = \sum_{t=0}^{\infty} \frac{\mathbb{E}[D_t]}{(1 + \bar{r}_S)^t}$$

A.4.5 Growth rate of Earnings/Dividends/Book Value

We know change in book value = retained earnings

$$\Delta BV_t = E_t \cdot \rho_t$$

substitute fact that: new earnings = ROE x BV

$$\Rightarrow BV_t - BV_{t-1} = ROE \cdot BV_{t-1} \cdot \rho_t$$

$$\Rightarrow \frac{BV_t - BV_{t-1}}{BV_{t-1}} = ROE \cdot \rho_t$$

Now define the growth rate in earnings as:

$$g = \frac{\Delta E_t}{E_{t-1}}$$

$$\Rightarrow g = \frac{E_t - E_{t-1}}{E_{t-1}}$$

substitute fact that: new earnings = ROE x BV

$$\Rightarrow g = \frac{ROE \cdot BV_t - ROE \cdot BV_{t-1}}{ROE \cdot BV_{t-1}}$$

$$\Rightarrow g = \frac{BV_t - BV_{t-1}}{BV_{t-1}}$$

Substitute above equation in red to get the desired result:

$$g = ROE \times \textbf{Plowback ratio}$$

A.5.3 CAPM by Lagrangian Optimisation (requires vector calculus)

DEFINITIONS

Let

$r_f =$ risk-free return
$r_P =$ return on our portfolio
$\sigma_P =$ standard deviation on our portfolio
$\mathbf{x} =$ portfolio weighting of n assets in the market, $(n \times 1)$
$\mathbf{r} =$ return of each of the n assets in the market, $(n \times 1)$
$Var(\mathbf{r}) = (n \times n)$ variance-covariance matrix of each asset's return
$\iota =$ vector of 1s, $(n \times 1)$

By definition, the return on our portfolio is the return on the proportion in the risk-free asset + the return on the proportion in risky assets:

$$r_P = \mathbf{x}'\mathbf{r} + r_f(1 - \mathbf{x}'\iota)$$

Taking expectations:

$$\mathbb{E}[r_P] = \mathbf{x}'\mathbb{E}[\mathbf{r}] + r_f(1 - \mathbf{x}'\iota)$$

The variance in the portfolio is just the covariance matrix times the square of weightings, since the risk-free asset has a variance of zero:

$$\sigma_P^2 = Var(r_P) = \mathbf{x}'Var(\mathbf{r})\mathbf{x}$$

SETUP

Our objective is to find the optimal portfolio, i.e. the lowest risk given return constraint:

$$\min_{\mathbf{x}}[\sigma_P]$$

s.t.

$$\mathbb{E}[r_P] = \mathbf{x}'\mathbb{E}[\mathbf{r}] + r_f(1 - \mathbf{x}'\iota)$$

STEP 1

Form Lagrangian:

$$\mathscr{L} = \sigma_P + \lambda\left[\mathbb{E}[r_P] - \mathbf{x}'\mathbb{E}[\mathbf{r}] - r_f(1 - \mathbf{x}'\iota)\right]$$

Take partials:

$$\mathscr{L}_{\mathbf{x}} = \sigma_P^{-1}Var(\mathbf{r})\mathbf{x} - \lambda\left[\mathbb{E}[\mathbf{r}] - r_f\iota\right] = 0 \quad \textbf{(eq.1)}$$

$$\mathscr{L}_{\lambda} = \mathbb{E}[r_P] - \mathbf{x}'\mathbb{E}[\mathbf{r}] - r_f(1 - \mathbf{x}'\iota) = 0$$

STEP 2

From our equilibrium analysis of the CAPM, we know a solution to this optimal portfolio weighting to be the market portfolio.

Introduce the market portfolio, with properties:

$$\sum x_i^M = 1, \text{(or alt. } \mathbf{x}'_M \iota = 1)$$

and so

$$r_M = \mathbf{x}'_M \mathbf{r}$$

$$\mathbb{E}[r_M] = \mathbf{x}'_M \mathbb{E}[\mathbf{r}]$$

$$\sigma_M^2 = Var(r_M) = \mathbf{x}'_M Var(\mathbf{r}) \mathbf{x}_M$$

Multiply both sides of first condition, (eq.1) by $\mathbf{x}'_M$

$$\Rightarrow \sigma_M^{-1} \mathbf{x}'_M Var(\mathbf{r}) \mathbf{x}_M - \lambda \left[\mathbf{x}'_M \mathbb{E}[\mathbf{r}] - r_f \mathbf{x}'_M \iota \right] = 0$$

Substitute market definitions above into the previous line to get:

$$\Rightarrow \sigma_M - \lambda \left[\mathbb{E}[\mathbf{r}_M] - r_f \right] = 0$$

$$\Rightarrow \frac{1}{\lambda} = \frac{1}{\sigma_M} \left[\mathbb{E}[r_M] - r_f \right] \quad \textbf{(eq.2)}$$

STEP 3

Now go back and rearrange eq.1 again to get:

$$\mathbb{E}[\mathbf{r}] = r_f \iota + \frac{1}{\lambda} \frac{Var(\mathbf{r}) \mathbf{x}_M}{\sigma_M}$$

Substitute eq.2 into the previous line:

$$\mathbb{E}[\mathbf{r}] = r_f \iota + \frac{Var(\mathbf{r}) \mathbf{x}_M}{\sigma_M^2} \left[\mathbb{E}[r_M] - r_f \right]$$

This is starting to look similar to our CAPM equation, instead in vector form now…

STEP 4

$$
\begin{aligned}
Var(\mathbf{r})\mathbf{x}_M &= \mathbb{E}\left[(\mathbf{r} - \mathbb{E}[\mathbf{r}])\,(\mathbf{r} - \mathbb{E}[\mathbf{r}])'\right]\mathbf{x}_M \\
&= \mathbb{E}\left[(\mathbf{r} - \mathbb{E}[\mathbf{r}])\left(\mathbf{x}_M'\mathbf{r} - \mathbf{x}_M'\mathbb{E}[\mathbf{r}]\right)'\right] \\
&= \mathbb{E}\left[(\mathbf{r} - \mathbb{E}[\mathbf{r}])\left(\mathbf{r}_M - \mathbb{E}[\mathbf{r}_M]\right)'\right] \\
&= Cov(\mathbf{r}, r_M)
\end{aligned}
$$

Hence substituting into the last line of step 3:

$$
\mathbb{E}[\mathbf{r}] = r_f \iota + \frac{Cov(\mathbf{r}, r_M)}{Var(r_M)}\left[\mathbb{E}[r_M] - r_f\right]
$$

STEP 5

Finally, let us define the vector, **beta** as:

$$
\boldsymbol{\beta} = \frac{Cov(\mathbf{r}, r_M)}{Var(r_M)}
$$

We can rewrite our previous line to get:

$$
\mathbb{E}[\mathbf{r}] = r_f \iota + \boldsymbol{\beta}\left[\mathbb{E}[r_M] - r_f\right]
$$

Hence, the return of an individual asset i in the market is:

$$
\mathbb{E}[r_i] = r_f + \beta_i\left[\mathbb{E}[r_M] - r_f\right]
$$

which is just the familiar CAPM equation.

A.7.4 Forward Price (no yield)

We use the Law of One Price to show the forward price under no arbitrage.

The forward payoff/value at time T is trivial, denoted f_T :

$$f_T = S_T - K$$

Now we discount the forward value at time T to time $t = 0$:

The present value of the stock is just S_0, while $PV(K) = Ke^{-rT}$
Hence

$$f_0 = S_0 - Ke^{-rT}$$

By definition, a forward contract requires no exchange at inception, hence $f_0 = 0$
Substituting definition for $f_0 = 0$ into previous line:

$$\Rightarrow 0 = S_0 - Ke^{-rT}$$

$$\Rightarrow K = S_0 e^{rT}$$

A.7.5 Forward Price (known income)

Similar to above, we use the Law of One Price to show the forward price under no arbitrage.

Now we have income that arrives during the course of the forward contract.
Denote:

$$I = \text{sum of PV of all income/dividends, hence } I = \sum_{t=0}^{T} PV(C_t)$$

The forward payoff/value at time T is trivial, denoted f_T :

$$f_T = S_T - FV(I) - K$$

Now we discount the forward value at time T to time $t = 0$:

The present value of the stock is just S_0, while $PV(K) = Ke^{-rT}$
Hence

$$f_0 = S_0 - I - Ke^{-rT}$$

By definition, a forward contract requires no exchange at inception, hence $f_0 = 0$
Substituting definition for $f_0 = 0$ into previous line:

$$\Rightarrow 0 = S_0 - I - Ke^{-rT}$$

$$\Rightarrow K = (S_0 - I)e^{rT}$$

A.7.6 Forward Price (known yield)

To demonstrate a different way of pricing forwards, we can set up a replicating portfolio, consisting of:

- borrow $Ke^{-r(T-t)}$
- $e^{-q(T-t)}$ units of the underlying

where:

$q =$ continous yield generated by the underlying asset

At maturity, the replicating portfolio will have a payoff of:

$$S_T e^{-qT} - K$$

which is the same as one forward contract.

Again, we use the Law of One Price to show the forward price under no arbitrage. The replicating portfolio and the forward must have the same setup cost, hence:

$$f_0 = S_0 e^{-qT} - Ke^{-rT}$$

At inception, nothing is exchanged, hence $f_0 = 0$. Substitute this into the previous line to get:

$$\Rightarrow K = S_0 e^{(r-q)T}$$

A.9.5 Option Price Bounds

Let us analyse the two possible payoffs at time T of the portfolio consisting of:

- 1 call option
- $Ke^{-r(T-t)}$ in cash

Hence:

$$\Pi_t = c_t + Ke^{-r(T-t)}$$

$$\Rightarrow \Pi_T = c_T + K$$

The two cases are $c_T = \begin{cases} \text{If } S_T > K \Rightarrow c_T = S_T - K \\ \text{If } S_T < K \Rightarrow c_T = 0 \end{cases}$

$$\Rightarrow \Pi_T = \begin{cases} \text{If } S_T > K \Rightarrow \Pi_T = S_T \\ \text{If } S_T < K \Rightarrow \Pi_T = K > S_T \end{cases}$$

In the best case scenario, the portfolio = stock price,
In the worst case scenario, the portfolio > stock price

Hence

$$\Pi_t \geq S_t$$

$$\Rightarrow c_t + Ke^{-r(T-t)} \geq S_t$$

$$\Rightarrow c_t \geq S_t - Ke^{-r(T-t)}$$

which is just our option price bound as required.

A.9.6 Put-Call Parity (no dividends)

Again, let

$c_t =$ value of call option at time t
$p_t =$ value of put option at time t
$S_t =$ stock price at time t
$K =$ strike price

Let us analyse the payoffs at time T, and using the LOOP (Law of One Price) to show the relationship must hold for all time.

Payoffs at time T:

$c_T = \max[S_T - K, 0]$
$p_T = \max[K - S_T, 0]$
$S_T = S_T$
and this fact, $PV_t(K) = Ke^{-r(T-t)}$ will come in handy later.

There are two cases to consider, either stock ends up higher or lower than the strike,

i.e. **CASE 1:** $S_T - K > 0$ **OR**
CASE 2: $S_T - K < 0$

CASE 1:
IF $S_T - K > 0$**,**
then: $c_T = S_T - K$
$p_T = 0$

$$\Rightarrow S_T + p_T + c_T = S_T + (0) - (S_T - K) = K$$

CASE 2:
IF $S_T - K < 0$**,**
then: $c_T = 0$
$p_T = K - S_T$

$$\Rightarrow S_T + p_T + c_T = S_T + (K - S_T) - (0) = K$$

Hence in all cases, the value is K at time T is just:

$$S_T + p_T - c_T = K$$

We discount K back to any time period t, hence we get:

$$\Rightarrow S_t + p_t - c_t = Ke^{-r(T-t)}$$

A.9.7 The Black-Scholes-Merton Model (Put price)

Given the formula for a call option, restated below:

$$c_0 = S_0 \cdot \Phi(d_1) - Ke^{-rT} \cdot \Phi(d_2)$$

where

$$d_1 = \frac{\ln(\frac{S_0}{K}) + T(r + \frac{\sigma^2}{2})}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

and the put-call parity relationship, also restated below:

$$S_t + p_t - c_t = Ke^{-r(T-t)}$$

$$\Rightarrow p_0 = Ke^{-rT} - S_0 + c_0$$

Substitute the call formula above into the put-call parity relationship:

$$\Rightarrow p_0 = [Ke^{-rT} - S_0] + [S_0 \cdot \Phi(d_1) - Ke^{-rT} \cdot \Phi(d_2)]$$

$$\Rightarrow p_0 = Ke^{-rT}\left[1 - \Phi(d_2)\right] - S_0\left[1 - \Phi(d_1)\right]$$

From the properties of the normal distribution, that is: $1 - \Phi(x) = \Phi(-x)$, we get:

$$p_0 = Ke^{-rT} \cdot \Phi(-d_2) - S_0 \cdot \Phi(-d_1)$$

again where

$$d_1 = \frac{\ln(\frac{S_0}{K}) + T(r + \frac{\sigma^2}{2})}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$