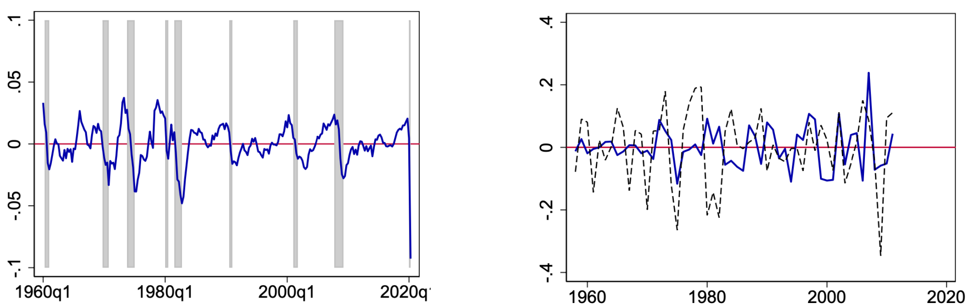
4 Production Networks and the Business Cycle

In previous macroeconomics courses, *aggregate shocks* are the source of business cycles. However, the macroeconomy is actually made up of hundreds of sectors, and millions of firms; which leads us to question:

- Is there really such a thing as an aggregate shock?
- Could shocks to individual firms or sectors drive business cycles? For example: decisions by firm's departments/managers, issues with shipments, inventories, strikes, lightning strike, earthquakes…

In this chapter, we will tackle

1. how to measure the effect of firm heterogeneity on the economy,
2. whether micro shocks explain relevant portions of aggregate fluctuations,
3. what amplifies the effect of microeconomic shocks.



(a) US aggregate

(b) by NAICS

Figure 4.1: Real Value Added (log deviations from Hodrick-Prescott filter) for the U.S. 1958-2011. For (b), dashed line is for NAICS 336111 (Automobiles), and solid for NAICS 325611 (Soap). *Source: NBER Manufacturing Database (2016)*

4.1 Diversification and Output Volatility

Consider the following economy with

1. a finite set, K consisting of production entities,
2. entity $k \in K$ is subject to idiosyncratic (IID) output $y_{k,t}$
3. factors are supplied inelastically; i.e. there are only productivity shocks.

In addition, we assume the growth in entity output has time-invariant variance σ_k, that is

$$\%\Delta y_{k,t} := \frac{y_{k,t} - y_{k,t-1}}{y_{k,t-1}} \tag{4.1}$$

$$\overset{\text{ass.}}{=} \sigma_k \varepsilon_{k,t} \tag{4.2}$$

where $\varepsilon_{k,t}$ is an IID, zero mean shock with variance one. Define *aggregate output* at time t as

$$Y_t = \sum_{k \in K} y_{k,t} \tag{4.3}$$

Lemma 4.1. *The variance of output growth, $\%\Delta Y_t$ can be shown to satisfy*

$$\sigma_Y^2 := \text{var}(\%\Delta Y_t) = \sum_{k \in K} \sigma_k^2 \left(\frac{y_{k,t-1}}{Y_{t-1}} \right)^2 \tag{4.4}$$

Proof. Output growth is

$$\%\Delta Y_t = \frac{Y_t - Y_{t-1}}{Y_{t-1}} \tag{4.5}$$

$$= \frac{1}{Y_{t-1}} \sum_{k \in K} (y_{k,t} - y_{k,t-1}) \tag{4.6}$$

$$= \frac{1}{Y_{t-1}} \sum_{k \in K} y_{k,t-1} \sigma_k \varepsilon_t \tag{4.7}$$

As ε_t is IID, the variance of the sum is the sum of variances; applying standard variance rules yields the result. □

4.1.1 Lucas (1977)'s diversification argument

If all firms have the same output variance, i.e. for all $k \in K$, $\sigma_k = \sigma \in \mathbb{R}_+$, and the same size, i.e. for all $k \in K$, $y_{k,t} = y_t$, then aggregate output growth volatility reduces to

$$\sigma_Y^2 = \frac{\sigma^2}{n} \tag{4.8}$$

where $n = |K|$ is the number of firms in K.

Empirically, n is large, hence the effect of shocks is small. This implies idiosyncratic shocks cannot explain macroeconomic fluctuations which due to diversification, means that individual shocks average out to 0. For example, one evidence across 472 NAICS 6-digit manufacturing industries estimates that $\hat{\sigma} = 0.09$, which implies

$$\hat{\sigma}_Y = \frac{0.09}{\sqrt{473}} = 0.0041 \tag{4.9}$$

although manufacturing GDP volatility is around 3%! If we look at the firm level instead (Axtell, 2001), with $N = 5.5$ million and $\hat{\sigma} = 0.12$ this implies

$$\hat{\sigma}_Y = \frac{0.12}{\sqrt{5.5 \times 10^6}} = 0.00005 \tag{4.10}$$

which is even worse. It is clear that we can conclude micro-shocks do not add up to explain aggregate fluctuations on their own.

4.1.2 Granular shocks, Gabaix (2011)

In most models, microeconomic shocks have no aggregate effects due to the assumption of a very large number of identical, small firms. This leads to idiosyncratic shocks canceling out by the law of large numbers, as we saw with Lucas (1977)'s diversification argument. However, in practice some firms are very large and can have large effects on output, such as Nokia in 2003 representing some 26% of Finnish private-sector GDP. In this section, we introduce Gabaix (2011)'s granular shocks idea, where we have large shocks to small entities (grains), not small macro shocks.

Starting with Lemma 4.1, here we assume instead that there is *heterogeneity in firm size* a_k but output volatility is identical, that is for all k, $\sigma_k = \sigma \in \mathbb{R}_+$. The following theorem leverages the definition of the Herfindahl–Hirschman Index (HHI) that we introduced in Chapter 1 to relate the number of firms to output.

Theorem 4.2. *Assuming homogeneous output volatility and a fat-tailed firm size distribution, aggregate output variance is decreasing in the number of firms,* $n = |K|$.

Proof. Aggregate output variance with homogeneous firm output volatility is

$$\sigma_Y^2 = \sigma^2 \sum_{k=1}^{n} \left(\frac{y_{k,t-1}}{Y_{t-1}} \right)^2 \tag{4.11}$$

$$= \sigma^2 H_{t-1} \tag{4.12}$$

where H_t is the Herfindahl–Hirschman Index at time t, defined as

$$H_t := \sum_{k=1}^{n} \left(\frac{y_{k,t}}{Y_t} \right)^2 \tag{4.13}$$

Gabaix (2011) shows that HHI declines slowly in n if the firm-size distribution is *fat tailed*, i.e. $\frac{\partial H_t}{\partial n} < 0$. By the chain rule of calculus,

$$\frac{\partial \sigma_Y^2}{\partial n} = \frac{\partial \sigma_Y^2}{\partial H_{t-1}} \frac{\partial H_{t-1}}{\partial n} \tag{4.14}$$

$$= \sigma^2 \frac{\partial H_{t-1}}{\partial n} < 0 \tag{4.15}$$

which is just the statement in the theorem. □

4.2 Production Networks

We follow a version of the model presented in Carvalho and Tahbaz-Salehi (2019).

4.2.1 Firms and production

Assume the economy consists of n competitive firms (or industries). Each firm, i uses a set of:

- *n intermediate inputs*, $\{x_{ij} : j = 1, \ldots, n\}$, or stacked as a vector $\mathbf{x}_i = (x_{i1}, \ldots, x_{in})^\top$,
- *m primary factors*, $\{l_{ij}; j = 1, \ldots, m\}$, or stacked as a vector $\mathbf{l}_i = (l_{i1}, \ldots, l_{im})^\top$, of which we assume an endowment vector $\mathbf{h} := (h_1, \ldots, h_m)^\top$.

Each firm has output y_i with Hicks-neutral production, and a production function $f : \mathbb{R}^{n+m} \to \mathbb{R}$ which satisfies the standard assumption of constant returns to scale (homogenous of degree one),

$$y_i = a_i f(\mathbf{x}_i, \mathbf{l}_i) \tag{4.16}$$

4.2.2 Households

We assume a *representative* household has preferences over all goods with consumption vector $\mathbf{c} := (c_1, \ldots, c_n)^\top$ and utility function, $u : \mathbb{R}^n \to \mathbb{R}$, which again we assume is homogeneous of degree one. The households problem is

$$\max \quad u(\mathbf{c}) \tag{4.17}$$
$$\text{s.t.} \quad \mathbf{p}^\top \mathbf{c} = \mathbf{w}^\top \mathbf{h} \tag{4.18}$$

where $\mathbf{w} \in \mathbb{R}^m$ is the factor price, and $\mathbf{p} \in \mathbb{R}^n$ is the price vector.

Lemma 4.3. *The households optimality condition implies the optimal consumption bundle,* $\mathbf{c}^*$ *satisfies*

$$\frac{u'(\mathbf{c}^*)}{u(\mathbf{c}^*)} = \frac{\mathbf{p}}{\text{GDP}} \tag{4.19}$$

where GDP $:= \mathbf{p}^\top \mathbf{c}^*$ *is the total value of goods consumed in equilibrium, and* ϕ *is the Lagrange multiplier associated with the household's problem in* (4.17)-(4.18).

Proof. The Lagrangian function associated with the household's problem is

$$\mathcal{L}(\mathbf{c}, \phi) = u(\mathbf{c}) + \phi\,(\mathbf{w}^\top \mathbf{h} - \mathbf{p}^\top \mathbf{c}) \tag{4.20}$$

The first order conditions imply

$$u'(\mathbf{c}^*) = \phi\,\mathbf{p} \tag{4.21}$$

where the derivative is understood in the denominator layout sense. Euler's homogeneous function theorem (see appendix) states that $[u'(\mathbf{c})]^\top \mathbf{c} = u(\mathbf{c})$. If we premultiply $\mathbf{c}$ by the transpose of (4.21) then we can apply Euler's theorem to the right hand side to obtain

$$u(\mathbf{c}^*) = \phi\,\mathbf{p}^\top \mathbf{c}^* = \phi\,\text{GDP} \tag{4.22}$$

Dividing the (4.21) by (4.22) yields the result. □

4.2.3 Social planner equilibrium

Consider a social planner that attempts to maximise utility in the model; they face the problem of maximising utility subject to goods clearing for each firm/product, and raw factors conforming to the endowment constraint. To describe the problem in matrix-vector form, we define the aggregate variables; let

- $\mathbf{X} := (\mathbf{x}_1, \cdots, \mathbf{x}_n) \in \mathbb{R}^{n\times n}$ represent the matrix of inputs for the economy with element x_{ij},
- $\mathbf{L} := (\mathbf{l}_1, \ldots, \mathbf{l}_n) \in \mathbb{R}^{m\times n}$ be the matrix of raw inputs into the economy, with element l_{ij}

For production, we have

- productivity vector $\mathbf{a} := (a_1, \ldots, a_n)^\top$,
- aggregate production function $F(\mathbf{X}, \mathbf{L}) := (f(\mathbf{x}_1, \mathbf{l}_1), \ldots, f(\mathbf{x}_n, \mathbf{l}_n))^\top \in \mathbb{R}^n$ that links with output $\mathbf{y} = (y_1, \ldots, y_n)^\top \in \mathbb{R}^n$ through

$$\mathbf{y} = \text{diag}(F(\mathbf{X}, \mathbf{L}))\,\mathbf{a} \tag{4.23}$$

The social planner's problem can therefore be expressed as

$$\max \quad u(\mathbf{c}) \tag{4.24}$$
$$\text{s.t.} \quad \mathbf{c} + \mathbf{X}^\top \mathbf{1}_n = \operatorname{diag}\left(F(\mathbf{X}, \mathbf{L})\right) \mathbf{a} \tag{4.25}$$
$$\mathbf{L}^\top \mathbf{1}_m = \mathbf{h} \tag{4.26}$$

An alternate presentation using summation and scalar variables is contained in the appendix.

Lemma 4.4. *The social planner's optimality conditions imply*

$$\frac{\partial \log u(\mathbf{c}^*)}{\partial \mathbf{a}} = \operatorname{diag}(F(\mathbf{X}, \mathbf{Z})) \frac{u'(\mathbf{c}^*)}{u(\mathbf{c}^*)} \tag{4.27}$$
$$\textit{and} \quad \frac{\partial \log u(\mathbf{c}^*)}{\partial \log a_i} = y_i \frac{\mathbf{e}_i^\top u'(\mathbf{c}^*)}{u(\mathbf{c}^*)} \tag{4.28}$$

where $\mathbf{e}_i$ *is the i-th basis vector, i.e.* $(0, \ldots, 1, \ldots, 0)^\top$ *with one at the i-th position.*

Proof (Matrix form). We derive each equation in turn.

1. The Lagrangian associated with the social planner's problem in (4.24)-(4.26) is

$$\mathcal{L} = u(\mathbf{c}) + \boldsymbol{\eta}^\top(\operatorname{diag}(F(\mathbf{X}, \mathbf{L}))\mathbf{a} - \mathbf{c} - \mathbf{X}^\top \mathbf{1}_n) + \omega^\top(\mathbf{L}^\top \mathbf{1}_m - \mathbf{h}) \tag{4.29}$$

   First order conditions imply

$$u'(\mathbf{c}^*) = \boldsymbol{\eta} \tag{4.30}$$

   where $\mathbf{c}^*$ is the optimal consumption bundle. On the other hand, by the Envelope Theorem,

$$\frac{\partial u(\mathbf{c}^*)}{\partial \mathbf{a}} = \operatorname{diag}(F(\mathbf{X}, \mathbf{L}))\boldsymbol{\eta} \tag{4.31}$$

   Using the optimality condition (4.30) above, we obtain

$$\frac{\partial u(\mathbf{c}^*)}{\partial \mathbf{a}} = \operatorname{diag}(F(\mathbf{X}, \mathbf{L}))\, u'(\mathbf{c}^*) \tag{4.32}$$

   Since $\frac{\partial \log u(\mathbf{c}^*)}{\partial \mathbf{a}} = \frac{1}{u(\mathbf{c}^*)} \frac{\partial u(\mathbf{c}^*)}{\partial \mathbf{a}}$, the result is obtained.

2. For the second result, note that $f(\mathbf{x}_i, \mathbf{l}_i) = y_i / a_i$; combine this with the first result to yield

$$\frac{\partial \log u(\mathbf{c}^*)}{\partial a_i} = \frac{y_i}{a_i} \frac{\mathbf{e}_i^\top u'(\mathbf{c}^*)}{u(\mathbf{c}^*)} \tag{4.33}$$

   Since $\frac{\partial \log u(\mathbf{c}^*)}{\partial a_i} = a_i \frac{\partial \log u(\mathbf{c}^*)}{\partial a_i}$, the result is obtained.

□

4.2.4 Hulten's Theorem

The following theorem is from Hulten (1978).

Theorem 4.5 (Hulten)**.** *Consider an efficient economy (welfare theorems apply, i.e. no distortions from price stickiness, market power, etc.). Then under minimal assumptions the following holds:*

$$\frac{\partial \log \text{GDP}}{\partial \log a_i} = \lambda_i := \frac{p_i y_i}{\text{GDP}} \tag{4.34}$$

where λ_i *is a firm/sector's Domar weight.*

We now prove Hulten's Theorem in the context of Carvalho and Tahbaz-Salehi's model.

Proof. In an efficient economy, the social planner equilibrium coincides with the households optimal choice. We combine them as follows; substitute the consumer's optimal condition (4.19) into the social planners optimal condition (4.28) to yield

$$\frac{\partial \log u(\mathbf{c}^*)}{\partial \log a_i} = y_i \frac{\mathbf{e}_i^\top \mathbf{p}}{\text{GDP}} \tag{4.35}$$

Using the fact that $\mathbf{e}_i^\top \mathbf{p} = p_i$ yields

$$\frac{\partial \log u(\mathbf{c}^*)}{\partial \log a_i} = \frac{p_i y_i}{\text{GDP}} \tag{4.36}$$

To complete the proof, we now argue that $u(\mathbf{c}^*) = \text{GDP}$. Recall that when discussing household conditions in (4.22), we showed that $u(\mathbf{c}^*) = \phi\,\text{GDP}$; to make $u(\mathbf{c}^*) = \text{GDP}$ hold true, it needs to be the case that $\phi = 1$. As utility functions are defined up to monotonic transforms, we can scale the utility function such that $\phi = 1$ holds. □

Hulten's Theorem implies that all one needs to know for effect of shocks to firm i is their sales share of GDP, and that the microeconomic details of production structure are irrelevant. The intuition is that sales capture both consumption and use as intermediate input. This statistic makes it straightforward to analyse macro implications of micro (e.g. sector-level) shocks, even when economy has a complex structure This is a powerful result and the foundation for most published aggregate TFP statistics (e.g. Bureau for Labor Statistics).

4.2.5 Combining Hulten and Gabaix

Corollary 4.6. *Assuming homogeneous productivity growth volatility of σ^2, and that Hulten's theorem holds, aggregate output growth variance can be expressed as*

$$\sigma_Y^2 = \sigma^2 H \tag{4.37}$$

where H is the Herfindahl–Hirschman Index.

Proof. The total derivative of log GDP is

$$\mathrm{d}\log \text{GDP} = \sum_{i=1}^{n} \frac{\partial \log \text{GDP}}{\partial \log a_i}\, \mathrm{d}\log a_i \tag{4.38}$$

Note that

$$\mathrm{d}\log \text{GDP} = \frac{\mathrm{d}\,\text{GDP}}{\text{GDP}}, \qquad \mathrm{d}\log a_i = \frac{\mathrm{d}a_i}{a_i} \tag{4.39}$$

Combined with (4.34) from Hulten's theorem yields

$$\frac{\mathrm{d}\,\text{GDP}}{\text{GDP}} = \sum_{i=1}^{n} \frac{p_i y_i}{\text{GDP}} \frac{\mathrm{d}a_i}{a_i} \tag{4.40}$$

Taking variances of both sides, and assuming independent and identical productivity growth variance of $\sigma^2 := \text{var}(\mathrm{d}\log a_i)$,

$$\sigma_Y^2 := \text{var}\left(\frac{\mathrm{d}\,\text{GDP}}{\text{GDP}}\right) = \sigma^2 \sum_{i=1}^{n} \left(\frac{p_i y_i}{\text{GDP}}\right)^2 \tag{4.41}$$

$$= \sigma^2 H \tag{4.42}$$

by the definition of the HHI, H. □

In **??** we see that the HHI using sales-to-GDP is around 0.07, and when multiplied by $\sigma = 0.12$ to yield σ_Y^2 in (4.37), we obtain a GDP volatility of around 1%, much closer to 3% than Lucas' original argument that was presented above.